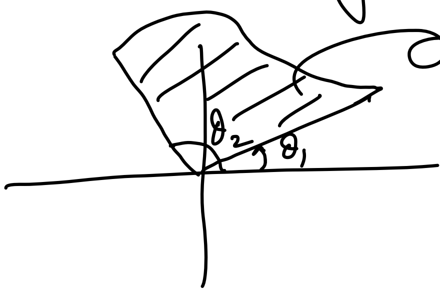
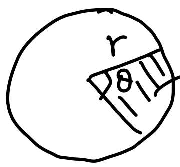
Given $r = f(\theta)$ in polar coord.



Area = ?

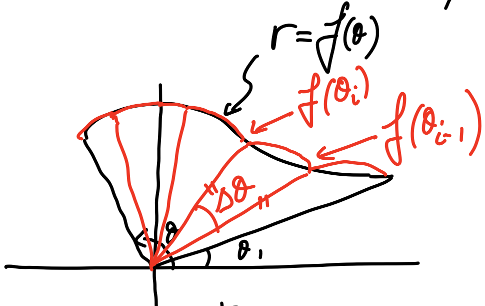


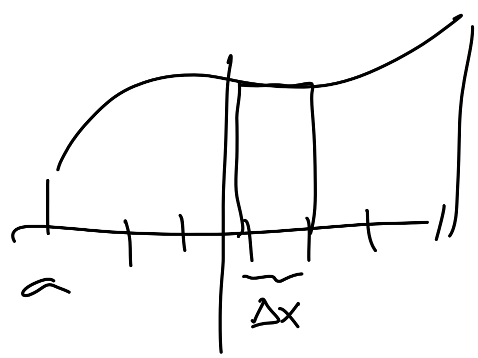
Area = ? : $\boxed{A = \frac{1}{2} r^2 \theta \ \{\theta \text{ in radian}\}}$

Area Theta.

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \frac{\cancel{\pi} r^2 \theta}{2\cancel{\pi}} = \underline{\frac{1}{2} r^2 \theta}$$





$$\text{Area} = \sum_{i=1}^{n} \frac{1}{2} r^2 \Delta\theta = \sum_{i=1}^{n} \frac{1}{2} \left(f(\theta_i)\right)^2 \Delta\theta$$

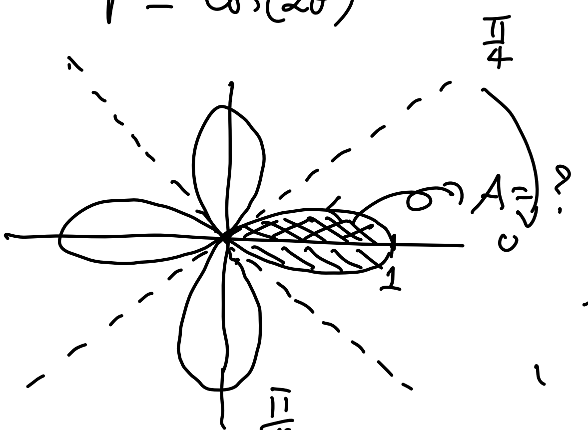
$$\text{Exact area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} \left(f(\theta_i)\right)^2 \Delta\theta$$

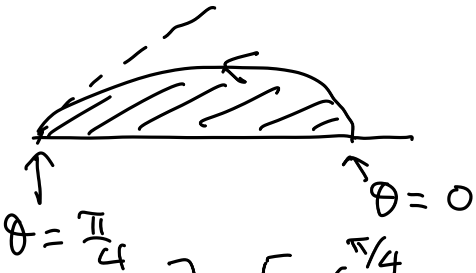
$$\boxed{A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta}$$

{ Remember this }

Ex: Find the area enclosed by one loop of the four-leave rose $r = \cos(2\theta)$.

$r = \cos(2\theta)$





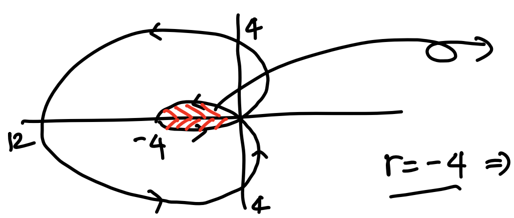
$$A = 2\left[\frac{1}{2}\int r^2\,d\theta\right] = \left[\cancel{\frac{1}{2}}\int_0^{\pi/4} (\cos(2\theta))^2\,d\theta\right]\cdot\cancel{2}.$$

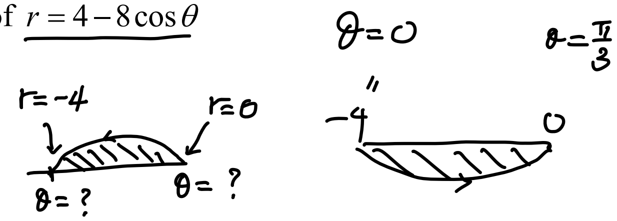
$$A = \int_0^{\frac{\pi}{4}} \cos^2(2\theta)\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos(4\theta))\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\right] = \boxed{\frac{\pi}{8}}$$

double angle formula.

Ex: Find the area of the inner loop of $r = 4 - 8\cos\theta$





$r = -4 \Rightarrow 4 - 8\cos\theta = -4$

$-8\cos\theta = -8 \Rightarrow \cos\theta = 1$

$\theta = 0$ ✓

$r = 0 \Rightarrow 4 - 8\cos\theta = 0$

$-8\cos\theta = -4 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ ✓

$$\text{Area} = \frac{1}{2}\int r^2\,d\theta = \cancel{2}\left[\cancel{\frac{1}{2}}\int_0^{\pi/3} (4 - 8\cos\theta)^2\,d\theta\right] = \int_0^{\pi/3}\left[16 - 64\cos\theta + 64\cos^2\theta\right]d\theta.$$

$$= \int_0^{\pi/3}\left[16 - 64\cos\theta + 32(1 + \cos(2\theta))\right]d\theta.$$

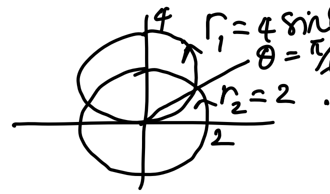
$$= 48\theta - 64\sin\theta + 16\sin(2\theta)\Big|_0^{\frac{\pi}{3}} = 16\pi - 64\cdot\frac{\sqrt{3}}{2} + 16\left[\frac{\sqrt{3}}{2}\right)$$

$$= \boxed{16\pi - 24\sqrt{3}}$$

sketch

Ex. Given $r_1 = 4\sin(\theta)$ and $r_2 = 2$. Set up integral(s) for area

a) Inside r1 / outside r2
b) Inside r2 / outside r1
c) Inside both r1 and r2.



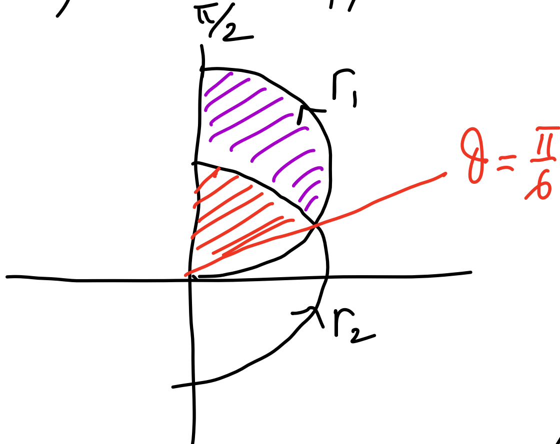
pts of intersection:

$r_1 = r_2$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

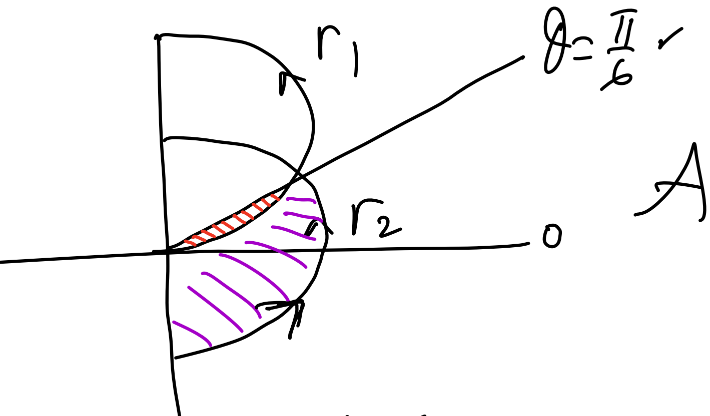
a) Inside r_1 / Outside r_2



$$A = \left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (4\sin\theta)^2 \, d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} 2^2 \, d\theta\right] \cdot 2$$

Inside r_1 Inside r_2

b) Inside r_2 / Outside r_1

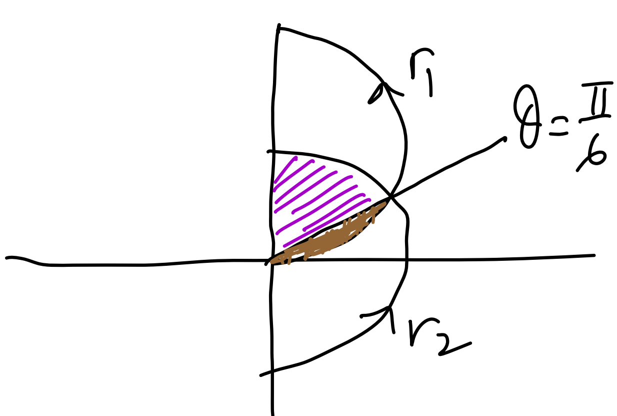


$$A = \left[\frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/6} 2^2 \, d\theta - \frac{1}{2}\int_{0}^{\pi/6} (4\sin\theta)^2 \, d\theta\right] \cdot 2$$

Inside r_2 Inside r_1

$\theta = ? = \cancel{\frac{3\pi}{2}}$; $-\frac{\pi}{2}$

c) Inside both r_1 & r_2



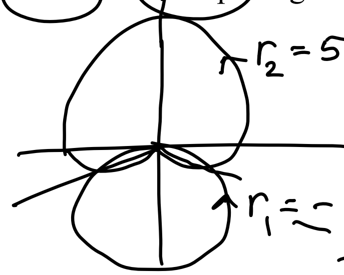
$$A = \left[\frac{1}{2}\int_{\pi/6}^{\pi/2} 2^2 \, d\theta + \frac{1}{2}\int_{0}^{\pi/6} (4\sin\theta)^2 \, d\theta\right] \cdot 2$$

Ex: Given $r_1 = -5\sin\theta$ *and* $r_2 = 5 + 5\sin\theta$. Sketch and set up integral(s) for area

a) Inside r1 / outside r2.

b) Inside r2 / outside r1

c) Inside both r1 and r2.

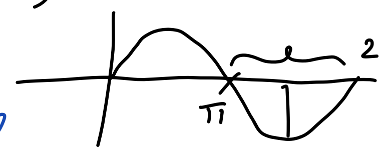


pts of intersection.

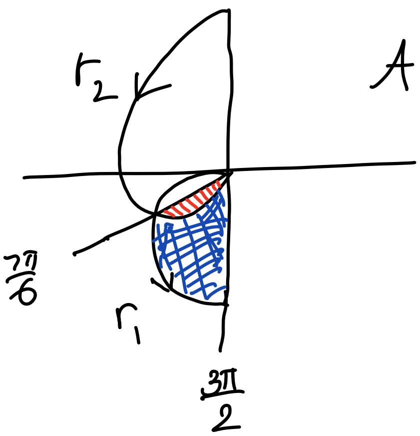
$r_1 = r_2$

$-5\sin\theta = 5 + 5\sin\theta$

$\sin\theta = -\frac{1}{2}$

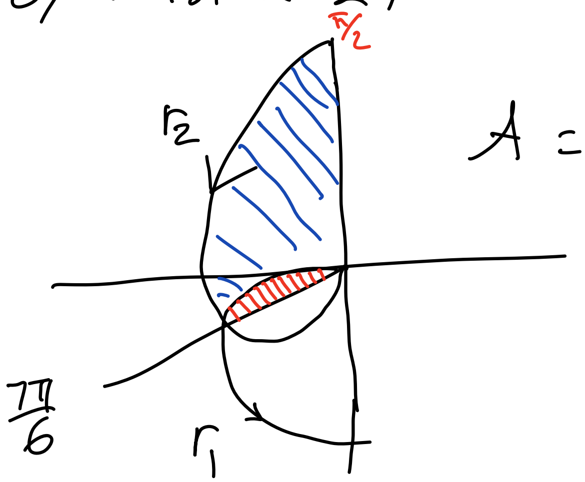


a) Inside r_1 / Outside r_2



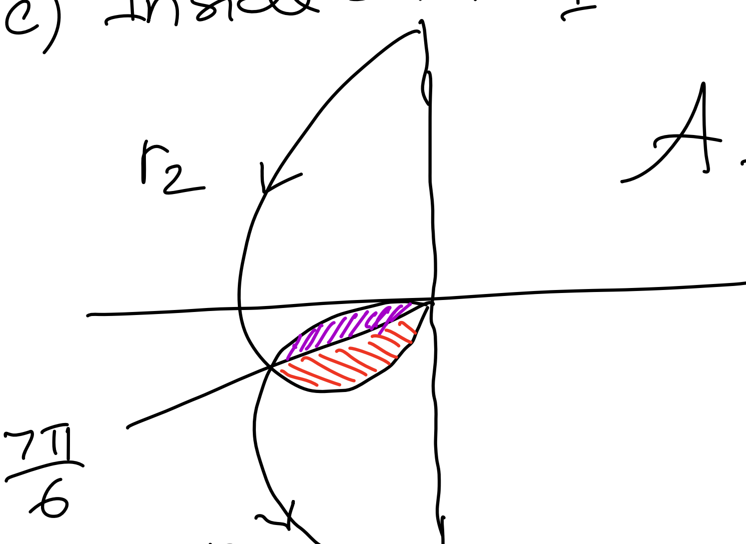
$$A = \left[\underbrace{\frac{1}{2}\int_{7\pi/6}^{3\pi/2} (-5\sin\theta)^2\,d\theta}_{\text{Inside } r_1} - \underbrace{\frac{1}{2}\int_{7\pi/6}^{3\pi/2} (5+5\sin\theta)^2\,d\theta}_{\text{Inside } r_2}\right]\cdot 2$$

b) Inside r_2 / outside r_1



$$A = \left[\frac{1}{2}\int_{\pi/2}^{7\pi/6} (5+5\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi}^{7\pi/6} (-5\sin\theta)^2\,d\theta\right]\cdot 2$$

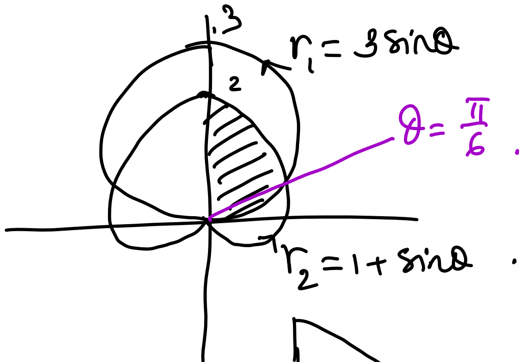
c) Inside both r_1 & r_2

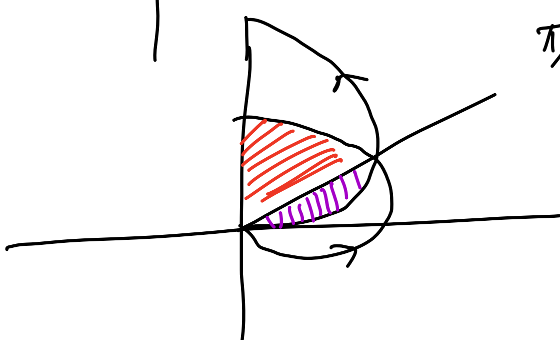


$$A = 2\left[\frac{1}{2}\int_{7\pi/6}^{3\pi/2} (5+5\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi}^{\pi/6} (-5\sin\theta)^2\,d\theta\right]$$

setup the integrals.

Ex: Find the area of the region that lies inside the circle $r_1 = 3\sin\theta$, and the cardioid $r = 1+\sin\theta$.





pts of intersection: $r_1 = r_2$

$3\sin\theta = 1+\sin\theta$

$2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$.

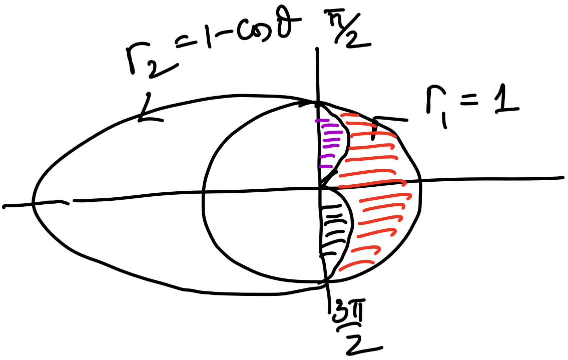
$\pi/6$

$$A = \left[\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\sin\theta)^2 d\theta + \frac{1}{2}\int_0^{\pi/6} 3^2 d\theta\right]\cdot 2$$

Ex: Sketch and find the area of the region that lies inside the circle $r_1 = 1$ and outside the cardiod $r_2 = 1-\cos\theta$

pts of intersection: $r_1 = r_2$

$1 = 1-\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$



$$A = \left[\frac{1}{2}\int_0^{\pi/2} 1^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2 d\theta\right]\cdot 2$$

Arc – Length for polar coordinate: of $r = f(\theta)$ where $\begin{cases} x = \underline{r\cos\theta} \\ y = \underline{r\sin\theta}. \end{cases}$

$$L = \int ds = \int^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$x = r\cos\theta \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = (r'\cos\theta - r\sin\theta)^2 = (r')^2\cos^2\theta \cancel{- 2r'r\cos\theta\sin\theta} + r^2\sin^2\theta.$$

$$y = r\sin\theta \Rightarrow \left(\frac{dy}{d\theta}\right)^2 = (r'\sin\theta + r\cos\theta)^2 = (r')^2\sin^2\theta \cancel{+ 2r'r\sin\theta\cos\theta} + r^2\cos^2\theta.$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (r')^2 + r^2$$

$$\Rightarrow L = \int_{\theta_1}^{\theta_2} \sqrt{(r')^2 + r^2}\, d\theta \quad \text{for } r = f(\theta) \text{ in polar Coord}$$

Note: $\underline{ds} =$
1. $\sqrt{1 + (y')^2}\, dx$ if $y = f(x)$
2. $\sqrt{1 + (x')^2}\, dy$ if $x = f(y)$
3. $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ if $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$
4. $\sqrt{(r')^2 + r^2}\, d\theta$ if $r = f(\theta)$ in polar coord.

Ex: Find the arc – length of the spiral $r = e^{\theta}$ *for* $0 \le \theta \le \pi$

$$L = \int ds = \int_0^{\pi} \sqrt{(r')^2 + r^2}\, d\theta \qquad \begin{cases} r = e^{\theta} \Rightarrow r^2 = (e^{\theta})^2 = e^{2\theta}. \\ r' = e^{\theta} \Rightarrow (r')^2 = (e^{\theta})^2 = e^{2\theta}. \end{cases}$$

$$L = \int_0^{\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \int_0^{\pi} \sqrt{2e^{2\theta}}\, d\theta =$$

$$= \sqrt{2} \int_0^{\pi} \sqrt{e^{2\theta}}\, d\theta = \sqrt{2} \int_0^{\pi} e^{\theta}\, d\theta$$

$$= \sqrt{2} \cdot e^{\theta} \Big|_0^{\pi} = \boxed{\sqrt{2} \cdot \left[e^{\pi} - 1\right]}$$

Ex: Find the arc – length of the cardioid $r = 1-\cos\theta$ for $0 \le \theta \le 2\pi$

$$L = \int ds = \int_0^{2\pi} \sqrt{(r')^2 + r^2}\, d\theta$$

$$\begin{cases} (r)^2 = (1-\cos\theta)^2 = 1 - 2\cos\theta + \cos^2\theta \\ (r')^2 = (\sin\theta)^2 = \sin^2\theta \end{cases} \quad \cos^2\theta + \sin^2\theta = 1$$

$$(r')^2 + r^2 = 1 - 2\cos\theta + 1 = 2 - 2\cos\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \sqrt{2}\int_0^{2\pi} \sqrt{\frac{1-\cos\theta}{1}} \cdot \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta \qquad \begin{cases} \text{let } u = 1+\cos\theta & \begin{cases} \theta = \pi \Rightarrow u = 0 \\ \theta = 0 \Rightarrow u = 2 \end{cases} \\ du = -\sin\theta\, d\theta \\ -du = \sin\theta\, d\theta \end{cases}$$

$$= 2\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta = 2\sqrt{2}\int_2^0 \frac{-du}{\sqrt{u}} = 2\sqrt{2}\int_0^2 u^{-1/2}\, du$$

$$= 2\sqrt{2} \cdot u^{1/2} \cdot 2 \Big|_0^2 = 2\sqrt{2} \cdot \sqrt{2} \cdot 2 = \boxed{8}$$

Area of a surface of revolution: Given $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$, and if the point $P:(r,\theta)$ traces trace the curve $r = f(\theta)$ exactly once for $\alpha \le \theta \le \beta$, then area of the surfaces generated revolving the curve about the x – and y – axes are given

1. Rotated about the x – axis. $S = \int_\alpha^\beta 2\pi r \boxed{\sin\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad for\ y \ge 0$

$\theta - y = r\sin\theta$

2. Rotated about the y – axis $S = \int_\alpha^\beta 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad for\ x \ge 0$

Ex: Find the area of the surface generated by revolving the right – hand loop of the lemniscate $r^2 = \cos(2\theta)$ about the y – axis.

$S = 2\pi \int r\, ds.$

$ds = \begin{cases} 1. \\ 2. \\ 3. \\ 4. \end{cases}$

$r = \begin{cases} x \rightarrow \text{about } y\text{-axis} \\ y \rightarrow \text{— } x\text{-axis} \end{cases}$