Chapter 11 Infinite Sequences and Series

Section 11.1 ***Sequences***

Def: A sequence can be thought as a list of numbers written in a definite order;

$\{a_1, a_2, a_3, \ldots, a_n, \ldots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Def: An infinite sequence (or sequence) of numbers is a function whose domain is the set of integers greater than or equal to some integer n.

Notation: $\{f(n)\} = \{f(1), f(2), \ldots, f(n), \ldots\} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$

Ex: $\boxed{\text{List the first 4 terms}}$ of the following sequences:

a) $\{a_n\} = \left\{\dfrac{2n+1}{n^2+3}\right\} = \left\{\dfrac{3}{4}, \dfrac{5}{7}, \dfrac{7}{12}, \dfrac{9}{19}, \ldots\right\}$

$a_1 = \dfrac{2(1)+1}{1^2+3} = \dfrac{3}{4}$ $a_2 = \dfrac{2(2)+1}{2^2+3} = \dfrac{5}{7}$

$a_3 = \dfrac{7}{12}$ $a_4 = \dfrac{9}{19}$

??

b) $\{b_n\} = \left\{\frac{(-1)^n}{(n+1)!}\right\} = \left\{-\frac{1}{2}, \frac{1}{6}, -\frac{1}{24}, \frac{1}{120}, \dots\right\}$

Alternating sequence

$b_1 = \frac{(-1)}{2!} = -\frac{1}{2}$

$b_2 = \frac{(-1)^2}{3!} = \frac{1}{6}$

$b_3 = \frac{(-1)^3}{4!} = -\frac{1}{24}$

$b_4 = \frac{(-1)^4}{5!} = \frac{1}{120}.$

---

c) $\{c_n\} = \{\cos(n\pi)\} = \{-1, 1, -1, 1, \dots\}$

$c_1 = \cos\pi = -1$

$c_2 = \cos(2\pi) = 1$

$c_3 = \cos(3\pi) = -1$

$c_4 = \cos(4\pi) = 1$

$\vdots$

---

d) $d_1 = 2$; $d_2 = -1$; $d_{n+2} = 2d_{n+1} + d_n - n!$

$d_1 = 2$

$d_2 = -1$

$d_3 = 2d_2 + d_1 - 1!$

$= 2(-1) + 2 - 1 = -1$

$d_4 = 2d_3 + d_2 - 2!$

$= 2(-1) + (-1) - 2 = -5.$

$\{d_n\} = \{2, -1, -1, -5, \dots\}$

General formula of a sequence:

Ex: Put the following sequence into its general formula

a) $\left\{\sqrt{2},\sqrt{3},\sqrt{4},\ldots,\sqrt{n},\ldots\right\} = \left\{\sqrt{n}\right\}_{n=2}^{\infty} = \left\{\sqrt{n+1}\right\}_{n=1}^{\infty} = \left\{\sqrt{n+2}\right\}_{n=0}^{\infty}$

b) $\left\{\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n}\ldots\right\} = \left\{\frac{1}{n}\right\}_{n=2}^{\infty} = \left\{\frac{1}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{n+2}\right\}_{n=0}^{\infty}$

c) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n-1}{n},\ldots = \left\{\frac{n-1}{n}\right\}_{n=2}^{\infty} = \left\{\frac{n}{n+1}\right\}_{n=1}$

d) $-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots,\cancel{(-1)^{n+1}\frac{1}{n}},\ldots = \left\{(-1)^{n+1}\cdot\frac{1}{n}\right\}_{n=2}$

e) $\left\{-\frac{2}{3},\frac{3}{9},-\frac{4}{27},\frac{5}{81},\ldots,\frac{(-1)^n(n+1)}{3^n}\right\} =$

$(-1)^1 = -$

f) $\left\{\frac{3}{5},-\frac{4}{25},\frac{5}{125},-\frac{6}{625},\frac{7}{3125},\ldots\right\} =$ $\left\{\frac{(n+2)(-1)^{n+1}}{5^{-n}}\right\}_{n=1}$

$5^{-1}\ 5^2\ 5^3\ 5^4\ 5^5 \ldots$

Ex: Recursive Formula: (Fibonacci Sequence)

$$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, ...\}$$

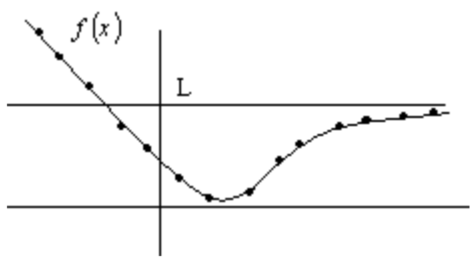
$$\left.\begin{array}{l} a_1 = 1 \\ a_2 = 1 \end{array}\right\} \quad a_n = a_{n-1} + a_{n-2} \text{ . for } n \geq 3 \text{ :}$$

Limit of a sequence: where does a sequence go to? i.e. what is the number (only one) that a sequence will be eventually approaches to.

Def: A sequence $\{a_n\}$ has the limit L and we write $\lim\limits_{n\to\infty} a_n = L$. If we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent)

A more precise version of limit:

A sequence $\{a_n\}$ has the limit L and we write $\lim\limits_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \varepsilon$ whenever n>N.



Def: Diverges to Infinity:

The sequence $\{a_n\}$ diverges to infinity if for every number M there is an integer N such that for all n larger than N , $a_n > n$. If this condition holds we write $\lim_{n\to\infty} a_n = \infty$

Theorem 1: $\lim_{n\to\infty} a_n = A$; $\lim_{n\to\infty} b_n = B$

$a)\quad \lim_{n\to\infty}(a_n \pm b_n) = A \pm B;$

$b)\quad \lim_{n\to\infty}(a_n b_n) = AB;$

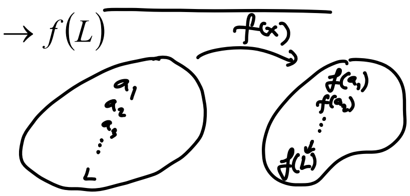
$c)\quad \lim_{n\to\infty}(k a_n) = kA;$

$d)\quad \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B};\ B \neq 0$

Squeeze Theorem : $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

$L \qquad L$

Theorem 3: Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \to f(L)$



Theorem 4: Suppose that $f(x)$ is a function defined for all $x \ge x_0$ and that $\{a_n\}$ is a sequence of real numbers such that $f(n) = a_n$ for all $n \ge n_0$. Then

$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$

Theorem 3: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then $\lim_{n\to\infty} a_n = L$

Methods:
- $\lim_{x\to\infty} \frac{\text{Poly}}{\text{Poly}} \div \frac{x^{\text{highest}}}{x^{\text{highest}}}$. ✓
- $\lim_{x\to\infty} \frac{5x^2-7x+4}{3x^2+1} \div \frac{x^2}{x^2} = \frac{5}{3}$
- L'H
- Squeeze thm.

Ex: Evaluate the limit of the following sequences:

a) $\{a_n\} = \left\{ \frac{2n^3 - 2n + 7}{\sqrt{9n^6 - 5n + 3}} \right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3 - 2n + 7}{\sqrt{9n^6 - 5n + 3}} \div \frac{n^3}{n^3} = \lim_{n\to\infty} \frac{2 - \frac{2}{n^2} + \frac{7}{n^3}}{\sqrt{9 - \frac{5}{n^5} + \frac{3}{n^6}}} = \boxed{\frac{2}{3}}$$

$\frac{2}{n^2} \to 0$, $\frac{7}{n^3} \to 0$, $\frac{5}{n^5} \to 0$, $\frac{3}{n^6} \to 0$

b) $\{b_n\} = \left\{ \left(1 + \frac{3}{n}\right)^{2n} \right\} \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 + \frac{3}{n}\right)^{2n} = 1^\infty$

$$\ln b_n = 2n \ln\left(1 + \frac{3}{n}\right)$$

$$\boxed{\lim_{n\to\infty} \ln b_n} = 2\lim_{n\to\infty} n \ln\left(1 + \frac{3}{n}\right) = \infty \cdot 0 = 2\lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{n}} = \frac{0}{0}$$

$$\stackrel{L'H}{=} 2\lim_{n\to\infty} \frac{\frac{1}{1+\frac{3}{n}} \cdot \left(-\cancel{\frac{3}{n^2}}\right)}{-\cancel{\frac{1}{n^2}}} = 6\lim_{n\to\infty} \frac{1}{1+\frac{3}{n}} = 6$$

$$\lim_{n\to\infty} \ln b_n = 6$$

$$\boxed{\lim_{n\to\infty} b_n = e^6}$$

c) $\{c_n\} = \left\{ \frac{\sin^2(2n+1)}{\sqrt{n^3+2}} \right\}$

$$\lim_{n\to\infty} c_n = ?$$

We know $0 \le \sin^2(2n+1) \le 1$

$$\underbrace{\lim_{n\to\infty} 0}_{=0} \le \lim_{n\to\infty} \frac{\sin^2(2n+1)}{\sqrt{n^3+2}} \le \underbrace{\lim_{n\to\infty} \frac{1}{\sqrt{n^3+2}}}_{=0}.$$

$\Rightarrow$ By the Squeeze Thm $\Rightarrow \lim_{n\to\infty} c_n = 0$.

d) $\{d_n\} = \{1 + (-1)^n\}$.

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \left(1 + (-1)^n\right)$$

$$= \underbrace{\lim_{n\to\infty} 1}_{} + \underbrace{\lim_{n\to\infty} (-1)^n}_{DNE.}$$

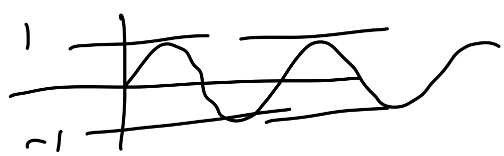
$$= 1 + DNE.$$

$$= \text{Divergent.}$$

---

e) $\{e_n\} = \{\sin(n)\}$

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} \sin(n) = DNE \Rightarrow \text{Divergent}$$



$5^n \to \infty$

$\left(-\frac{1}{5}\right)^n \to 0$

Note: $\left.r^{n\to\infty}\right\} \Rightarrow \lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ \text{Divergent} & \text{if } |r| \geq 1 \end{cases}$

f) $\{a_n\} = \left\{\dfrac{9^n + 7^n}{9^n + 5^n}\right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{9^n + 7^n}{9^n + 5^n} \div \frac{9^n}{9^n}.$$

$$= \lim_{n\to\infty} \frac{1 + \left(\frac{7}{9}\right)^n \to 0}{1 + \left(\frac{5}{9}\right)^n \to 0.} = \boxed{1}$$

g) $\{a_n\} = \{\ln(n+2) - \ln(3n+2)\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\ln(n+2) - (\ln(3n+2))\right] = \infty - \infty.$$

$$= \lim_{n\to\infty} \ln\left(\frac{n+2}{3n+2}\right) = \ln\left[\lim_{n\to\infty} \frac{n+2}{3n+2}\right]$$

$$= \boxed{\ln\left(\frac{1}{3}\right)}$$

h) $\{a_n\} = \left\{\dfrac{n!}{n^n}\right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{n^n}.$$

$$0 \le \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} \le \frac{1\cdot\cancel{n}\cdot\cancel{n}\cdots\cancel{n}}{n\cdot\cancel{n}\cdot\cancel{n}\cdots\cancel{n}}.$$

$$0 \le \frac{n!}{n^n} \le \frac{1}{n}.$$

$$\underbrace{\lim_{n\to\infty} 0}_{=0} \le \lim_{n\to\infty}\frac{n!}{n^n} \le \underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}$$

By the Squeeze Thm.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n!}{n^n} = 0$$

g) $\{a_n\} = \{\sqrt[n]{n}\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \infty^0$$

$$\ln a_n = \ln n^{\frac{1}{n}} = \frac{1}{n}\ln n = \frac{\ln n}{n}.$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty}\frac{\ln n}{n} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1.$$

Note: For what values of r is the sequence $\{r^n\}$ convergence?

Sol: $\lim_{n\to\infty} r^n = \begin{cases} \infty \text{ if } r>1 \\ 1 \text{ if r} = 1 \\ 0 \text{ if } 0 < \text{r} < 1 \end{cases}$ Demonstrate this by plotting point for n.

$-1 < r < 1$

Theorem: The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & \text{if r} = 1 \\ 0 & \text{if -1} < \text{r} < 1 \end{cases}$$

Monotone and Bounded Sequences:

Def: Let $\{a_n\}$ be a sequence of real numbers.

- The sequence is monotone increasing if $a_n \le a_{n+1}$ for all $n \ge 1$
- The sequence is monotone decreasing if $a_n \ge a_{n+1}$ for all $n \ge 1$
- The sequence is bounded above if there is a number M such that $a_n \le M$ for all $n \ge 1$
- The sequence is bounded below if there is a number m such that $a_n \ge M$ for all $n \ge 1$

(If a sequence is bounded above and bounded below, we say that the sequence is bounded. If a sequence is not bounded, we say that it is unbounded.)

Ex: a) For positive integer n, let $a_n = \sqrt{n^4 + n^3} - n^2$. Show that the sequence $\{a_n\}$ is monotone increasing and unbounded.

For sequences $\Rightarrow$
1. Divide top & bot. by $n^{\text{highest power}}$.
2. Squeeze Thm. ✓
3. L'H. ✓

Ex: Find the limit of $\{a_n\}$.

a) $\{a_n\} = \left\{ \dfrac{\sqrt[3]{8n^6+5n-2}}{7n^2+n-10} \right\}$.

$\frac{5n}{n^6} = \frac{5}{n^5}$

sol: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\sqrt[3]{8n^6+5n-2}}{7n^2+n-10} \div \dfrac{n^2}{n^2} = \lim_{n\to\infty} \dfrac{\sqrt[3]{8 + \frac{5}{n^5} - \frac{2}{n^6}}}{7 + \frac{1}{n} - \frac{10}{n^2}}$

$\frac{5}{n^5} \to 0$, $\frac{2}{n^6} \to 0$, $\frac{1}{n} \to 0$, $\frac{10}{n^2} \to 0$

$$= \frac{\sqrt[3]{8}}{7} = \boxed{\frac{2}{7}} \text{ convergent.}$$

b) ~~Let $\{a_n\} = \left\{\frac{(-1)^n}{n}\right\}$. Show that the sequence $\{a_n\}$ is bounded but not monotone.~~

b) $$\{a_n\} = \left\{ \frac{3n^2 + 7n - 5}{\sqrt[3]{2n^5 + n^3 - 1}} \right\}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3n^2 + 7n - 5}{\sqrt[3]{2n^5 + n^3 - 1}} \div \frac{n^2}{n^2} = \lim_{n \to \infty} \frac{3 + \underbrace{\frac{7}{n}}_{\to 0} - \underbrace{\frac{5}{n^2}}_{\to 0}}{\sqrt[3]{\underbrace{\frac{2}{n}}_{0} + \underbrace{\frac{1}{n^3}}_{0} - \underbrace{\frac{1}{n^6}}_{0}}} = \frac{3}{0^+} = \infty$$

$\Rightarrow$ Divergent.

c) Show that the sequence $a_n = \frac{3}{n+5}$ is monotone decreasing.

d) Show that the sequence $a_n = \frac{n}{n^2 + 1}$ is monotone decreasing.

c) $\{a_n\} = \left\{ \dfrac{3}{n+5} \right\}$ ➔ it's monotone decreasing.

d) $\{a_n\} = \left\{ \dfrac{n}{n^2+1} \right\}$

<u>The Monotone Sequence Theorem</u>: Let $\{a_n\}$ be a monotone increasing sequence of real numbers.

a) if $\{a_n\}$ is bounded above, then $\lim_{n\to\infty} a_n$ exists.

b) if $\{a_n\}$ is not bounded above, then $\lim_{n\to\infty} a_n = \infty$

Ex: Define a sequence $\{a_n\}$ by the recursion relationship $a_1 = 1$; $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. Show that the sequence converges and find its limit.

Ex: Investigate the sequence $\{a_n\}$ defined by the recursive definition

$$a_1 = 2; a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for n = 1, 2, 3,...}$$

Know these.

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ 2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$ 3. $\lim_{n\to\infty} x^{1/n} = 1$ (x>0)

4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$ 5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any x) 6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

Ex: Evaluate the following limits:

a) $\lim_{n\to\infty} \frac{\ln(n^3)}{4n} = \lim_{n\to\infty} \left(\frac{3\ln n}{4n}\right) = \frac{3}{4} \underbrace{\lim_{n\to\infty} \frac{\ln n}{n}}_{=0} = \boxed{0}$

b) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2 = \left(\lim_{n\to\infty} \sqrt[n]{n}\right)^2 = 1^2 = \boxed{1}$

c) $\lim_{n\to\infty} \left(\frac{3^n}{n^3}\right)$

d) $\lim_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}} =$

e) $\lim_{n\to\infty}(n+4)^{1/(n+4)}$

$$\lim_{n\to\infty} \frac{5\cdot 7^{n+1} + 2\cdot 9^n + 3\cdot 4^n}{2\cdot 9^{n+1} - 4\cdot 7^n + 3^n} \div \frac{9^n}{9^n}$$

$$= \lim_{n\to\infty} \frac{35\cdot\cancel{\left(\frac{7}{9}\right)^n} + 2 + 3\cdot\cancel{\left(\frac{4}{9}\right)^n}}{18 - 4\left(\frac{7}{9}\right)^n + \left(\frac{3}{9}\right)^n} = \frac{2}{18}$$

$$= \boxed{\frac{1}{9}}$$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & |r|<1 \\ 1 & r=1 \\ \text{divergent} & |r|>1 \end{cases}$$

f) $\lim_{n\to\infty}\frac{(10/11)^n}{(9/10)^n+(11/12)^n} \div \frac{\left(\frac{11}{12}\right)^n}{\left(\frac{11}{12}\right)^n}$.

$r<1$

$$= \lim_{n\to\infty} \frac{\left(\frac{10}{11}\cdot\frac{12}{11}\right)^n}{\left(\frac{9}{10}\cdot\frac{12}{11}\right)^n + 1} = \lim_{n\to\infty} \frac{\cancel{\left(\frac{120}{121}\right)^n} \to 0}{\cancel{\left(\frac{108}{110}\right)^n} + 1} = \frac{0}{0+1} = \frac{0}{1} = \boxed{0}$$

g) $\lim_{n\to\infty}\frac{3^n 6^n}{2^{-n} n!} = \lim_{n\to\infty} \frac{3^n\cdot 6^n\cdot 2^n}{n!} = \lim_{n\to\infty}\frac{36^n}{n!} = 0$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \text{ for all } x$$

h) $\lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n = \lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\div\frac{3n}{3n}\right)^n$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{3n}\right)^n}{\left(1-\frac{1}{3n}\right)^n} = \frac{e^{1/3}}{e^{-1/3}} = e^{2/3}.$$

$$\left(\left(\ldots\right)^{3n+4}\right)$$

$$\{a_n\} = \left\{ \left(\frac{5n-2}{5n+3} \right)^{3n+4} \right\}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{5n-2}{5n+3} \div \frac{5n}{5n} \right)^{3n+4}$$

$$= \left[\lim_{n\to\infty} \frac{\left(1 - \frac{2}{5n}\right)^{n}}{\left(1 + \frac{3}{5n}\right)^{n}} \right]^{3} \cdot \left[\lim_{n\to\infty} \frac{5n-2}{5n+3} \right]^{4}$$

$$\lim_{n\to\infty} \frac{5n-2}{5n+3} \div \frac{n}{n} = \frac{5}{5} = 1$$

$$= \left[\frac{e^{-2/5}}{e^{3/5}} \right]^{3} \cdot (1)^{4} = \left(\frac{1}{e} \right)^{3} = \boxed{\frac{1}{e^{3}}}$$

$$\boxed{\lim_{n\to\infty} \left(1 + \frac{x}{e} \right)^{n} = e^{x}}$$