Series

Def: Given a sequence $\{a_n\} = \{\overbrace{a_1, a_2, a_3, a_4, \dots}^{\ell}\}$

A series is the sum of all entries of a sequence.

$$\text{Series} = a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{i=1}^{\infty} a_i = \sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} a_k$$

$\Rightarrow$ Limit of a series $=$ Sum as $n \to \infty$.
$= S$

Sequence of **Partial Sum**: Sum $S = \underbrace{a_1} + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$\vdots$
$S_n = a_1 + a_2 + \cdots + a_n.$
$\vdots$

$\{S_1, S_2, S_3, S_4, \dots\} = \{S_n\}_{n=1}^{\infty}$

$$\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n \quad \text{where } S_n = a_1 + a_2 + \cdots + a_n}$$

Def: Given a series $\sum_{n=1}^{\infty} a_n$, let $S_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_n} = \sum_{i=1}^{n} a_i$ be the n[th] partial sum of the series. If the sequence $\{S_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$\sum a_n = S$

$\sum_{n=1}^{\infty} a_n = S$, the number S is called the sum of the series. Otherwise, the series is called divergent.

Geometric Series $\sum_{n=1}^{\infty} ar^{n-1} = S = \lim_{n\to\infty} S_n$.

$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ \text{div.} & \text{if } |r|\geq 1 \end{cases}$

Where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

multiply by r

$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}}$

$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + ar^n$.

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$S_n = \dfrac{a(1-r^n)}{1-r}$.

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a(1-\lim_{n\to\infty} r^n)}{1-r}$.

$$\boxed{S = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{divergent} & \text{if } |r|\geq 1 \end{cases}}$$ G.S.

Note: $\sum_{n=1}^{\infty} a\cdot r^{n-1} = \sum_{n=0}^{\infty} a\cdot r^n = \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{div.} & \text{if } |r|\geq 1 \end{cases}$

$(\#)^{\text{power of } n} \Rightarrow$ G.S.

Ex: Evaluate the following:

a) $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} a\cdot r^{n-1}$ $\begin{cases} a = 3 \\ r = \left|\frac{1}{2}\right| < 1 \end{cases}$

$\Rightarrow$ It's convergent by G.S.

$\Rightarrow$ Sum $= S = \dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{2}} = \boxed{6}$

b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \left(-\frac{1}{2}\right)^{n-1} + \ldots = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$. $\begin{cases} a = 1 \\ r = \left|-\frac{1}{2}\right| < 1 \end{cases}$

$\Rightarrow$ It's convergent by G.S. $\Rightarrow S = \dfrac{a}{1-r} = \dfrac{1}{1+\frac{1}{2}} = \boxed{\frac{2}{3}}$

c) $\frac{\pi}{2} + \frac{\pi^2}{4} + \frac{\pi^3}{8} + \ldots = \sum_{n=1}^{\infty} \left(\frac{\pi}{2}\right)^{n-1} =$ $\begin{cases} a = 1 \\ r = \left|\frac{\pi}{2}\right| > 1 \end{cases}$

$\Rightarrow$ It's divergent by G.S.

d) $\frac{5}{1} - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots = \sum_{n=1}^{\infty} \frac{5\cdot 2^{n-1}(-1)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 5\cdot\left(-\frac{2}{3}\right)^{n-1}$ $\begin{cases} a = 5 \\ r = \left|-\frac{2}{3}\right| < 1 \end{cases}$

($5 = 5\cdot 2^0$, $10 = 5\cdot 2$, $20 = 5\cdot 2^2$, $40 = 5\cdot 2^3$; $1 = 3^0$, $3 = 3^1$, $9 = 3^2$, $27 = 3^3$)

$\Rightarrow$ It's convergent $\Rightarrow S = \dfrac{a}{1-r} = \dfrac{5}{1+\frac{2}{3}} = \dfrac{15}{3+2} = \boxed{3}$ ✓

e) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n+2}} = \sum_{n=1}^{\infty} a \cdot r^{n-1}$

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n-1} \cdot 4^3} = \sum_{n=1}^{\infty} \frac{1}{64}\left(\frac{3}{4}\right)^{n-1} \quad \begin{cases} a = \frac{1}{64} \\ r = \left|\frac{3}{4}\right| < 1 \end{cases}$$

$\Rightarrow$ It's convergent by G.S $\Rightarrow$ $S = \frac{a}{1-r} = \frac{1/64}{1-3/4} \cdot \frac{64}{64} = \frac{1}{64-48} = \boxed{\frac{1}{16}}$

f) $\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^{2n+1}} = \sum_{n=0}^{\infty} a\, r^n$

$$= \sum_{n=0}^{\infty} \frac{3^n \cdot 3^{-1}}{4^n \cdot 2} = \sum_{n=0}^{\infty} \frac{1}{6}\left(\frac{3}{4}\right)^n \quad \begin{cases} a = \frac{1}{6} \\ r = \left|\frac{3}{4}\right| < 1 \end{cases}$$

$\Rightarrow$ It's convergent by G.S. $S = \frac{a}{1-r} = \frac{1/6}{1-\frac{3}{4}} \cdot \frac{12}{12} = \frac{2}{12-9} = \boxed{\frac{2}{3}}$

g) $\sum_{n=0}^{\infty} \frac{2^{3n-1}}{7^{n+1}} = \sum_{n=0}^{\infty} a \cdot r^n$

$2^{3n} = (2^3)^n = 8^n$.

$$= \sum_{n=0}^{\infty} \frac{2^{3n} \cdot 2^{-1}}{7^n \cdot 7} = \sum_{n=0}^{\infty} \frac{1}{14}\left(\frac{8}{7}\right)^n \quad \begin{cases} a = \frac{1}{14} \\ r = \left|\frac{8}{7}\right| > 1 \end{cases}$$

$\Rightarrow$ It's divergent by G.S.

h) $$\sum_{n=3}^{\infty} \frac{2^{n+1}}{3^{n-2}} = \sum_{n=0}^{\infty} \frac{2^{n+3+1}}{3^{n+3-2}} = \sum_{n=0}^{\infty} \frac{2^n \cdot 2^4}{3^n \cdot 3} = \sum_{n=0}^{\infty} \frac{16}{3}\left(\frac{2}{3}\right)^n \quad \begin{cases} a = \frac{16}{3} \\ r = \left|\frac{2}{3}\right| < 1 \end{cases}$$

1. -3 or

$n=1$, $\boxed{n=0}$

$$\sum_{n=1} \frac{2^{n+2+1}}{3^{n+2-2}}$$

$\|$

$\Rightarrow$ Convergent by G.S. : $S = \frac{a}{1-r}$

$S = \frac{16/3}{1-2/3} = \boxed{16}$

$$\sum_{n=1} \frac{2^{n-1}\cdot 2^4}{3^{n-1}\cdot 3} = \sum_{n=1} \left(\frac{16}{3}\right)\left(\frac{2}{3}\right)^{n-1}$$

Ex: Determine equivalent fraction:
$5.\overline{23} = \frac{a}{b}$.

$$\boxed{2.\overline{43}} \quad ? \quad \frac{a}{b}$$

$2.\overline{75}$

$$5.\overline{23} = 5.23232323\ldots$$

$$= 5 + 0.23 + 0.0023 + 0.000023 + 0.00000023 + \cdots$$

$$= 5 + \underbrace{\frac{23}{10^2} + \frac{23}{10^4} + \frac{23}{10^6} + \frac{23}{10^8} + \cdots}$$

$$= 5 + \sum_{n=1}^{\infty} \frac{23}{10^{2n}} = \sum a\cdot r^{n-1} = S$$

$$\sum_{n=1} \frac{23}{100^{n-1}\cdot 100} = \sum_{n=1}^{\infty} \frac{23}{100}\left(\frac{1}{100}\right)^{n-1} \quad \begin{cases} a = \frac{23}{100} \\ r = \left|\frac{1}{100}\right| < 1 \end{cases}$$

$$S = \frac{a}{1-r} = \frac{23/100}{1-\frac{1}{100}} = \frac{23}{99}.$$

$$5.\overline{23} = 5 + S = 5 + \frac{23}{99} = \frac{5(99)+23}{99} = \boxed{\frac{518}{99}}$$

Def: A telescoping series is one in which the nth term can be expressed in the form
$a_n = b_n - b_{n+1}$

T.S.

Convergence of a telescoping series:

If $\sum_{n=1}^{\infty} a_n$ is a telescoping series with $a_n = b_n - b_{n+1}$ then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $\{b_n\}$ converges. Furthermore, if $\{b_n\}$ converges to L, then $\sum_{n=1}^{\infty} a_n$ converges to L

$$\boxed{\sum a_n = S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(a_1+a_2+\cdots+a_n)}$$

Ex: Find the sum of the following:

a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+7n+12} = \sum_{n=1}^{\infty}\frac{1}{(n+3)(n+4)} = \sum_{n=1}^{\infty}\left(\frac{A}{n+3}+\frac{B}{n+4}\right)$

$$A\Big|_{n=-3} = \frac{1}{-3+4} = 1 \quad ; \quad B\Big|_{n=-4} = \frac{1}{-4+3} = -1$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n+3}-\frac{1}{n+4}\right) = S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{4}-\frac{1}{n+4}\right) = \boxed{\frac{1}{4}}$$

where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$$= \underbrace{\left(\frac{1}{4}-\cancel{\frac{1}{5}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{6}}\right)}_{n=2} + \underbrace{\left(\cancel{\frac{1}{6}}-\cancel{\frac{1}{7}}\right)}_{n=3} + \cdots + \underbrace{\left(\cancel{\frac{1}{n+3}}-\frac{1}{n+4}\right)}_{n=n}$$

$$= \frac{1}{4} - \frac{1}{n+4}$$

b) $\displaystyle\sum_{n=0}^{\infty}\underbrace{\left(e^{1/(n+3)}-e^{1/(n+2)}\right)}_{a_n} = S = \lim_{n\to\infty} S_n.$

where $S_n = a_0 + a_1 + a_2 + \cdots + a_n$

$$= \underbrace{\left(\cancel{e^{\frac{1}{3}}}-e^{\frac{1}{2}}\right)}_{n=0} + \underbrace{\left(\cancel{e^{\frac{1}{4}}}-\cancel{e^{\frac{1}{3}}}\right)}_{n=1} + \underbrace{\left(\cancel{e^{\frac{1}{5}}}-\cancel{e^{\frac{1}{4}}}\right)}_{n=2} + \cdots + \underbrace{\left(e^{\frac{1}{n+3}}-\cancel{e^{\frac{1}{n+2}}}\right)}_{n=n}$$

$$S_n = -e^{\frac{1}{2}} + e^{\frac{1}{n+3}}.$$

$$S = \lim S_n = \lim_{n\to\infty}\left(-e^{\frac{1}{2}} + e^{\frac{1}{n+3}}\right) = e^0 - e^{\frac{1}{2}}$$

$$= \boxed{1-\sqrt{e}}$$

c) $\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3} = \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)} = \sum_{n=1}^{\infty}\left(\frac{A}{n+1}+\frac{B}{n+3}\right)$

$A\Big|_{n=-1} = \frac{1}{-1+3} = \frac{1}{2}$; $B\Big|_{n=-3} = \frac{1}{-3+1} = -\frac{1}{2}$.

$\sum_{n=1}^{\infty}\left(\frac{1/2}{n+1}-\frac{1/2}{n+3}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\overbrace{\left(\frac{1}{n+1}-\frac{1}{n+3}\right)}^{a_n} = \frac{1}{2}S = \frac{1}{2}\lim_{n\to\infty}S_n$.

where $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$= \underbrace{\left(\frac{1}{2}-\frac{1}{4}\right)}_{n=1} + \underbrace{\left(\frac{1}{3}-\frac{1}{5}\right)}_{n=2} + \underbrace{\left(\frac{1}{4}-\frac{1}{6}\right)}_{n=3} + \underbrace{\left(\frac{1}{5}-\frac{1}{7}\right)}_{n=4} + \underbrace{\left(\frac{1}{6}-\frac{1}{8}\right)}_{n=5} + \cdots + \underbrace{\left(\frac{1}{n+1}-\frac{1}{n+3}\right)}_{n=n}$

$S_n = \frac{1}{2}+\frac{1}{3}-\frac{1}{n+3} \Rightarrow S = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+3}\right) = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}$.

Ans: $\frac{1}{2}\left(\frac{5}{6}\right) = \boxed{\frac{5}{12}}$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

D.T.T. (Divergent Test Theorem) "only to prove "divergent""

The test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

D.T.T. $\sum a_n$ if $\lim_{n\to\infty} a_n = \begin{cases} \neq 0 \\ \text{DNE} \end{cases}$.

Then $\sum a_n$ is divergent

i.e If $\lim_{n\to\infty} a_n = 0$

$\Rightarrow$ Inconclusive

Ex: Show that the following series is divergent.

a) $\sum_{n=1}^{\infty}\cos(n\pi) \stackrel{\text{let}}{=\!=} a_n$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\cos(n\pi) = \text{DNE} \neq 0$

$\therefore$ By D.T.T. $\sum a_n$ is divergent.

b) $\sum_{n=1}^{\infty}\frac{5n^3+2n-7}{2n^3+1} \stackrel{\text{let}}{=\!=} a_n$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5n^3+2n-7}{2n^3+1} = \frac{5}{2} \neq 0$

$\therefore$ By D.T.T. $\Rightarrow \sum a_n$ is divergent.

1. G.S.
2. T.S.
3. D.T.T.

⋮

11.

Theorem: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series

$$\sum_{n=1}^{\infty} ca_n \quad \text{and} \quad \sum_{n=1}^{\infty}(a_n \pm b_n)$$

Ex: Find the sum of the series

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3^{n-1}}+\frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty}\frac{1}{3^{n-1}} + \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

$$= \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}}_{\begin{cases} a=1 \\ r=\left|\frac{1}{3}\right|<1 \end{cases}} + \sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right) \qquad \begin{cases} A\big|_{n=0} = \frac{1}{0+1} = 1 \\ B\big|_{n=-1} = \frac{1}{-1} = -1 \end{cases}$$

$$= \qquad + \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{n\to\infty} S_n$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$
$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$= 1-\frac{1}{n+1}$$

$$= \frac{1}{1-\frac{1}{3}} + \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = \frac{1}{2/3}+1 = \frac{3}{2}+1 = \boxed{\frac{5}{2}}$$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{\cos(n\pi)+2^{n+1}}{5^n}$

Note: $\cos(n\pi) = \underset{n=0}{1}, -1, 1, -1, 1, -1 \dots = (-1)^n$.

$$\sum_{n=0}^{\infty}\frac{(-1)^n+2^{n+1}}{5^n} = \underbrace{\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n}_{\substack{a=1 \\ r=\left|-\frac{1}{5}\right|<1}} + \underbrace{\sum_{n=0}^{\infty}2\left(\frac{2}{5}\right)^n}_{\substack{a=2 \\ r=\left|\frac{2}{5}\right|<1}}$$

$$\Rightarrow = \frac{a}{1-r}+\frac{a}{1-r} = \frac{1}{1+\frac{1}{5}}+\frac{2}{1-\frac{2}{5}}.$$

$$= \frac{5}{6}+\frac{10}{3} = \boxed{\frac{25}{6}}$$

11.3 The Integral Test and Estimates of Sums

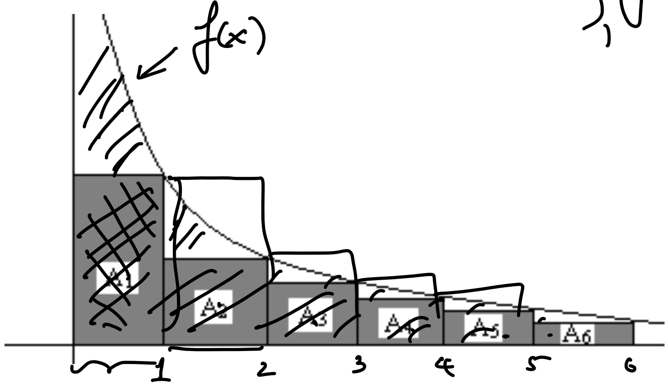
(Integral Test Thm: I.T.T.)

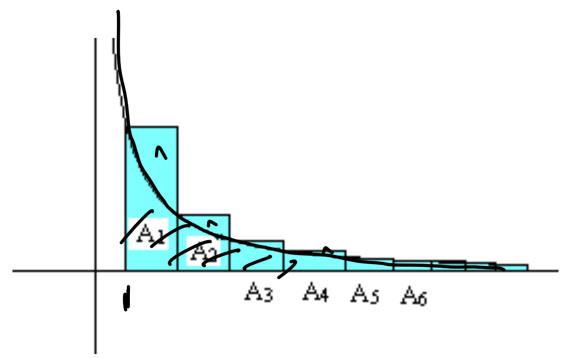
Given a series $\sum_{n=1}^{\infty} a_n$ and consider $f(x) = a_x$ (*i.e. replaced n by x*)

$$\sum_{n=1}^{\infty} a_n = 1\cdot a_1 + 1a_2 + 1a_3 + \cdots$$

$$\int_1^{\infty} f(x)dx \geq \sum_{n=1}^{\infty} a_n \Rightarrow \int_1^{\infty} f(x)dx \text{ is convergent} \xrightarrow{\text{then}} \sum_{n=1}^{\infty} a_n \text{ is also convergent}$$

$\int_1^{\infty} f(x)dx$ is divergent $\Rightarrow \sum a_n$ is also divergent.





Case I

Case II

Introducing the test be given a series $\sum_{n=1}^{\infty} a_n$ and define a function $f(n) = a_n$

Case I: $\sum_{n=1}^{\infty} a_n \leq \int_1^{\infty} f(x)dx \Rightarrow$ If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Case II: $\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)dx \Rightarrow \int_1^{\infty} f(x)dx$ is divergent, and then $\sum_{n=1}^{\infty} a_n$ is divergent.

I.T.T. ***The Integral Test***: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. Suppose that $a_n = f(n)$ $\rightarrow a_x = f(x)$ where f is continuous, positive, decreasing function of x for all $x \geq N$ for some positive integer N. Then the series $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ both converge or both diverge.

Review p – test theorem: $\int_{a>0}^{\infty} \frac{1}{x^p}dx$ is $\begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p\le 1 \end{cases}$

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \overset{\text{let}}{=} a_n.$

~~G.S. #1~~ ~~T.S.~~ ~~D.T.T~~

Define $f(x) = a_x = \frac{1}{x\sqrt{x}}$. $\begin{cases} 1.\ \text{continuous} \\ 2.\ \text{positive}. \\ 3.\ \text{decreasing} \end{cases}$

Consider $\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x\cdot x^{1/2}}dx = \int_1^{\infty} \frac{1}{x^{3/2}}dx$ $\begin{cases} p=\frac{3}{2}>1 \text{ is convergent} \\ \text{by P-Test}. \end{cases}$

$\therefore$ by I.T.T. $\sum a_n$ is also convergent.

b) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \overset{\text{let}}{=} a_n \Rightarrow f(x) = a_x = \frac{1}{x\sqrt{\ln x}}$ $\begin{cases} 1.\ \text{cont.} \\ 2.\ \text{positive} \\ 3.\ \text{decreasing} \end{cases}$ over $[2,\infty)$

Consider: $\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx$.

Let $u=\ln x$, $du = \frac{1}{x}dx$

$= \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\cdot du = \int_{\ln 2}^{\infty} \frac{1}{u^{1/2}}du$ $\begin{cases} p=\frac{1}{2}<1 \text{ is} \\ \text{divergent} \\ \text{by P-Test} \end{cases}$

$\therefore$ by I.T.T. $\Rightarrow \sum a_n$ is also divergent.

P – test for series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent $\int \frac{1}{x^p}$

Sol: Let $f(x) = \frac{1}{x^p} \Rightarrow \int_1^{\infty} \frac{1}{x^p}dx$ is convergent if $p>1$ and it's divergent if $p \le 1$

$\boxed{\text{The p-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent if } p>1 \text{ and it's divergent if } p\le 1}$

$\int_{a>0}^{\infty} \frac{1}{x^p}dx = \sum_{n=1}^{\infty} \frac{1}{n^p}$: $\begin{cases} p>1 \Rightarrow \text{Convergent} \\ p\le 1 \Rightarrow \text{divergent} \end{cases}$

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty}\frac{1}{(5n)^3} = \sum_{n=1}^{\infty}\frac{1}{125\cdot n^3} = \frac{1}{125}\sum_{n=1}^{\infty}\frac{1}{n^3}$ → $p=3>1$ is convergent by p-Test.

b) $\sum_{n=1}^{\infty}\frac{2}{\sqrt[4]{n^3}} = 2\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$ $\left\{ \begin{array}{l} p=\frac{3}{4}<1 \text{ is} \\ \text{divergent by p-Test.} \end{array} \right.$

$\left\{ \begin{array}{l} \text{1. G.S.} \\ \text{2. T.S.} \\ \text{3. D.T.T.} \\ \text{4. I.T.T.} \\ \text{5. p-Test.} \end{array} \right.$

c) $\sum_{n=1}^{\infty}\frac{\ln n}{n}$

G.S. $S = \frac{a}{1-r} = \sum a_n$

T.S. $S = \lim_{n\to\infty} S_n = \sum a_n$

Estimate the sum of a series: Given a "convergent" series $\sum_{n=1}^{\infty} a_n$ — Approximate the sum S.

$$S = \sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{R_n}$$

$$= S_n + R_n.$$

$$S = S_n + R_n \qquad \begin{cases} S_n = \text{approximation of the sum } S. \\ R_n = \text{Error}. \end{cases}$$

$S \approx S_n$

$S = S_n + R_n$

$|R_n| = |S - S_n|$

Remainder Estimate for the integral test:

Suppose $f(n) = a_n$, where f is continuous, positive, decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$ $\Rightarrow$ $\boxed{\text{Error} = |R_n| \leq \int_n^{\infty} f(x)dx}$

only for I.T.T.

_Ex_: a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx S_{10} = a_1 + a_2 + a_3 + \cdots + a_{10}$$

$$= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} = \#$$

$\frac{1}{n^3} = a_n$

$$\text{Error} = R_n = R_{10} \leq \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty} \frac{1}{x^3}dx = \lim_{t\to\infty} \int_{10}^{t} x^{-3}dx$$

$$= \lim_{t\to\infty} \frac{x^{-2}}{-2}\Big|_{10}^{t} = -\frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{t^2} - \frac{1}{10^2}\right]$$

$\frac{1}{t^2} \to 0$

$$= -\frac{1}{2}\left(-\frac{1}{10^2}\right) = \frac{1}{200} = 0.005 = 0.5\%$$

b) How many terms are required to ensure that the sum is accurate to within 0.0005?

$$\underline{n = ?} \quad \text{such that} \quad \text{Error} = |R_n| < 0.0005$$

$$\text{where} \quad |R_n| \le \int_n^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_n^{t} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \frac{x^{-2}}{-2}\Big|_n^{t}$$

$$= -\frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{t^2} - \frac{1}{n^2}\right] = \frac{1}{2n^2} < 0.0005 \Rightarrow \frac{1}{2(0.0005)} < n^2$$

$$\frac{1}{t^2} \to 0$$

$$n^2 > \frac{1}{0.001} = 1000 \Rightarrow n > \sqrt{1000} = 31.62.$$

$$\boxed{n \ge 32 \text{ terms.}}$$

Ex: Determine how many terms are needed to ensure the error within 0.005.

$$\sum_{n=1}^{\infty} \frac{1}{n\left[\ln(3n)\right]^3}$$

$$n = ? \quad \text{so that error} < 0.005.$$

$$\text{Error} = |R_n| \le \int_n^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_n^{t} \frac{1}{x\left[\ln(3x)\right]^3}\,dx$$

$$\left\{ \begin{array}{l} \text{let } u = \ln(3x) \\ du = \frac{1}{3x}\cdot 3\cdot dx = \frac{1}{x}dx \end{array}\right.$$

$$= \lim_{t\to\infty}\int \frac{du}{u^3} = \lim_{t\to\infty}\int u^{-3}\,du = \lim_{t\to\infty}\frac{u^{-2}}{-2}$$

$$= -\frac{1}{2}\lim_{t\to} \frac{1}{(\ln(3x))^2}\Big|_n^{t} = -\frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{\left[\ln(3t)\right]^2} - \frac{1}{\left[\ln(3n)\right]^2}\right]$$

$$\frac{1}{\left[\ln(3t)\right]^2} \to 0$$

$$\text{Error} = \frac{1}{2\left[\ln(3n)\right]^2} < 0.005.$$

$$\text{Trial \& error} \Rightarrow n = \underline{20} \Rightarrow \frac{1}{2\left[\ln(60)\right]^2} = 0.0298$$

$$n = 200 \Rightarrow \frac{1}{2\left[\ln(600)\right]^2} = 0.0122$$

$$= 0.0047 < 0.005$$

$\boxed{n = 10{,}000} \Rightarrow \dfrac{\quad}{2\left[\ln(30{,}000)\right]^2} = 0.00\ldots$

Ex: How many terms are needed to ensure the approximation accurate with 0.00001

$$\sum_{n=1}^{\infty} \frac{n}{[3n^2+1]^4} \; ; \qquad n = ? \text{ so that error} \le 0.00001$$

Sol:

$$\text{Error} = |R_n| \le \int_n^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_n^t \frac{x}{[3x^2+1]^4}\,dx$$

let $u = 3x^2+1 \Rightarrow du = 6x\,dx$

$\dfrac{du}{6} = x\,dx.$

$$= \lim_{t\to\infty} \int \frac{\frac{1}{6}\,du}{u^4} = \frac{1}{6} \lim_{t\to\infty} \int u^{-4}\,du.$$

$$= \frac{1}{6} \lim_{t\to\infty} \frac{u^{-3}}{-3} = -\frac{1}{18} \lim_{t\to\infty} \frac{1}{(3x^2+1)^3} \Bigg|_n^t.$$

$$= -\frac{1}{18} \lim_{t\to\infty} \left[\underbrace{\frac{1}{(3t^2+1)^3}}_{\to 0} - \frac{1}{(3n^2+1)^3} \right]$$

$$= \frac{1}{18(3n^2+1)^3}$$

$$\text{Error} = |R_n| < \frac{1}{18(3n^2+1)^3} < 0.00001$$

$$10^{-5}$$

Trial & error pick $n = 5$

$$\frac{1}{18(3 \cdot 25 + 1)^3} = 0.000,000,127 < 0.00001$$

$\cancel{n = 2}$