1. G.S.
2. T.S.
3. D.T.T.
4. I.T.T.
5. p-Test.

Direct. ***The Comparison Tests***

<u>The Comparison Test</u>: (D.C.T.T.) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

$\sum \text{small} \le \sum \text{big}$.

a) $\sum$ big is convergent $\Rightarrow$ $\sum$ small is convergent

b) $\sum$ small is divergent $\Rightarrow$ $\sum$ big is divergent.

Note: for convergence $\Rightarrow$ Construct bigger.
for divergence $\Rightarrow$ " smaller

a) $$\sum_{n=1}^{\infty} \frac{7n^2 + 5n^3 + 4\cdot 1}{\sqrt{\cancel{4}n^{12} + \cancel{5n+4}}} \le \sum_{n=1}^{\infty} \frac{7n^3 + 5n^3 + 4n^3}{\sqrt{n^{12}}} = \sum_{n=1}^{\infty} \frac{16n^3}{n^6} = 16 \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Dominant Terms: $\frac{n^3}{\sqrt{n^{12}}} = \frac{n^3}{n^6} = \frac{1}{n^{3>1}} \Rightarrow$ Convergent $\Rightarrow$ Construct bigger.

$p = 3 > 1 \Rightarrow$ is convergent by p-Test.

$\therefore$ by D.C.T.T. $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is also convergent.

b) $$\sum_{n=1}^{\infty} \frac{4n^3 + 5n + 8}{\sqrt{n^8 + 5n + 12}} \geq \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^8 + 5n^8 + 12n^8}} = \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{18n^8}} = \frac{1}{\sqrt{18}} \sum_{n=1}^{\infty} \frac{n^3}{n^4} = \frac{1}{\sqrt{18}} \sum_{n=1}^{\infty} \frac{1}{n}.$$

Dominant terms: $\frac{n^3}{\sqrt{n^8}} = \frac{n^3}{n^4} = \frac{1}{n} \rightarrow$ divergent $\Rightarrow$ construct smaller.

$p = 1 \Rightarrow$ divergent by p-Test.

$\therefore$ by D.C.T.T. $\Rightarrow$ $\sum a_n$ is also divergent.

c) $\sum_{n=1}^{\infty} \frac{n^2 + \sin(7n+\pi)}{n^5 + 3n + 5} \leq \sum_{n=1}^{\infty} \frac{n^2+n^2}{n^5} = \sum_{n=1}^{\infty} \frac{2n^2}{n^5} = 2\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\left\{ \begin{array}{l} p=3>1 \text{ is} \\ \text{convergent} \\ \text{by } p\text{-Test.} \end{array} \right.$

Dominant terms: $\frac{n^2}{n^5} = \frac{1}{n^3}$ $\rightarrow$ convergent $\Rightarrow$ Construct bigger.

$\therefore$ by D.C.T.T. $\Rightarrow$ $\sum a_n$ is also convergent.

$n^2 < 4^n$

d) $\sum_{n=1}^{\infty} \frac{n^2 + 4^n + 4}{n^5 \cdot 4^{n+1}} \leq \sum_{n=1}^{\infty} \frac{4^n + 4^n + 4\cdot 4^n}{n^5 \cdot 4^{n+1}} = \sum_{n=1}^{\infty} \frac{6 \cdot \cancel{4^n}}{n^5 \cdot \cancel{4^n} \cdot 4} = \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n^5}$ $\left\{ \begin{array}{l} p=5>1 \text{ is} \\ \text{convergent by} \\ p\text{-Test.} \end{array} \right.$

Dominant terms: $\frac{\cancel{4^n}}{n^5 \cdot 4^{n+1}} = \frac{1}{4n^5}$ $\rightarrow$ convergent $\Rightarrow$ Construct bigger.

$\therefore$ by D.C.T.T. $\Rightarrow$ $\sum a_n$ is also convergent.

e) $\sum_{n=1}^{\infty} \frac{\cancel{2}n^2 \cancel{+ 3\cos^2(n) + 4}}{\sqrt{n^5 + 5n + 7 \cdot 1}} \geq \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5 + 5n^5 + 7n^5}} = \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{13n^5}} = \frac{1}{\sqrt{13}}\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}-2}} = \frac{1}{\sqrt{13}}\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

Dominant terms: $\frac{n^2}{\sqrt{n^5}} = \frac{n^2}{n^{5/2}} = \frac{1}{n^{\frac{5}{2}-2}} = \frac{1}{n^{\frac{1}{2}}}$ $<1 \rightarrow$ divergent $\rightarrow$ Construct smaller

$p = \frac{1}{2} < 1 \Rightarrow \left\{ \begin{array}{l} \text{divergent by} \\ p\text{-Test.} \end{array} \right.$

$\therefore$ by D.C.T.T. $\Rightarrow$ $\sum a_n$ is also divergent.

The Limit Comparison Test: L.C.T.T.

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a ***finite number*** and $c > 0$, then either both series converge or both diverge.

Suppose $a_n > 0$ *and* $b_n > 0$ for all $n \ge N$

1. $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \Rightarrow$ both converge or both diverge.
2. $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges*
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges*

When + & − signs mixed in the problems.

Ex: Test for convergence

a) $\sum_{n=1}^{\infty} \frac{7n^3 - n + 4}{\sqrt{49n^{10} - 9n + 4}} = a_n.$

let b_n = dominant terms $= \frac{n^3}{\sqrt{n^{10}}} = \frac{n^3}{n^5} = \frac{1}{n^2} = b_n.$

Consider $c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[\frac{7n^3 - n + 4}{\sqrt{49n^{10} - 9n + 4}} \cdot \frac{n^2}{1}\right] = \frac{7}{\sqrt{49}} = 1$

and $\sum b_n = \sum \frac{1}{n^2}$ $\{p = 2 > 1$ convergent by p-Test.

$\therefore$ by L.C.T.T. $\Rightarrow \sum a_n$ is also convergent.

1d)

$$\sum_{n=1}^{\infty} \frac{7}{n(2n-1)} = a_n$$

$b_n = \frac{7}{n^2}$

$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{7}{n(2n-1)} \cdot \frac{n^2}{7}$

$= \lim \frac{n^2}{2n^2-n} = \frac{1}{2} > 0$

$\sum b_n = \sum \frac{7}{n^2}$ is conv.

$\therefore$ by L.C.T.T. $\sum a_n$ is convergent

b) $$\sum_{n=1}^{\infty} \frac{7n-n+4}{\sqrt{2n^4+5n-2}} \stackrel{\text{let}}{=} a_n.$$

Consider b_n = dominant terms $= \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}$

Consider $c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\overset{6n+4}{7n-n+4}}{\sqrt{2n^4+5n-2}} \cdot \frac{n}{1} = \frac{6}{\sqrt{2}} > 0.$

$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-Test.

$\therefore$ by L.C.T.T. $\Rightarrow \sum a_n$ is also divergent.

c) $$\sum_{n=1}^{\infty} \frac{1}{2^n-1}; \stackrel{\text{let}}{=} a_n$$

b_n = dominant terms of $a_n = \frac{1}{2^n} = b_n$

Consider $c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{2^n-1} \cdot \frac{2^n}{1}\right) = \lim_{n\to\infty} \left(\frac{2^n}{2^n-1} \div \frac{2^n}{2^n}\right) =$

$\lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}\to 0} = 1 > 0$

$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n \to r = \left|\frac{1}{2}\right| < 1$

$\Rightarrow$ is convergent by G.S.

$\therefore$ by L.C.T.T. $\sum a_n$ is convergent

d) $$\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$$

d) $$\sum_{n=1}^{\infty} \frac{n+1}{n\cdot 2^n} \le \sum_{n=1}^{\infty} \frac{n+n}{n\cdot 2^n} = \sum \frac{2\cancel{n}}{\cancel{n}\cdot 2^n} = \sum \frac{1}{2^{n-1}}$$

Dominant term: $\frac{\cancel{n}}{\cancel{n}\cdot 2^n} = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \rightarrow$ Convergent $\Rightarrow$ Construct bigger.

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \quad \begin{cases} a = 1 \\ r = \left|\frac{1}{2}\right| < 1 \end{cases}$$

is convergent by G.S.

$\therefore$ By D.C.T.T. $\sum a_n$ is also convergent.

$$\lim_{n\to\infty}\left(\boxed{1} + \frac{x}{n}\right)^{n} = e^{x}$$

$$\{a_n\} = \left\{ \left(\frac{2n+1}{2n-3}\right)^{3n-4} \right\}$$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2n+1}{2n-3}\right)^{3n-4}$$

$$= \lim_{n\to\infty} \left[\frac{2n+1}{2n-3} \div \frac{2n}{2n}\right]^{3n-4}$$

$$= \left[\lim_{n\to\infty} \frac{\left(1+\frac{1}{2n}\right)^n}{\left(1-\frac{3}{2n}\right)^n}\right]^3 \left[\lim_{n\to\infty} \frac{2n+1}{2n-3}\right]^{-4}$$

$$= \left[\frac{e^{1/2}}{e^{-3/2}}\right]^3 (1)^{-4}$$

Video

$$= \left(e^2\right)^3 = \boxed{e^6}$$