Alternating Series (ALT)

Def: An alternating series is a series of the form $a_1 + a_2 - a_3 + a_4 - a_5 + \ldots$
Such as $1+2-3+4-5+6\ldots$

The Alternating Series Test: Let $\sum_{n=1}^{\infty} \cancel{(-1)^{n-1}} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots \quad (a_n > 0)$

Satisfies the following conditions:

Decreasing
i) $\boxed{a_{n+1} \le a_n} \Rightarrow$ Decreasing without negative sign.
ii) $\boxed{\lim\limits_{n\to\infty} a_n = 0}$

Then the series is convergent.

(Note: if a_n satisfies the 2 cond. $\Rightarrow \sum (-1)^n a_n$ is convergent
Otherwise $\Rightarrow$ It's inconclusive)

Ex: Determine whether the following series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} \longrightarrow$ let $a_n = \frac{1}{n}$.

$(-1)^n$
$(-1)^{n-1}$

i.) $a_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = a_n$ ✓

ii.) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$ ✓

By ALT $\Rightarrow \sum (-1)^{n-1} a_n$ is convergent.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ this is ~~an alternating series, but~~ $\cancel{\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0}$

$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{3n}{4n-1} \overset{\text{let}}{=\!=} a_n$.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3n}{4n-1} \overset{\text{let}}{=\!=} b_n$.

$\lim\limits_{n\to\infty} b_n = \underbrace{\lim\limits_{n\to\infty} (-1)^n}_{\text{DNE}} \cdot \underbrace{\lim\limits_{n\to\infty} \left(\frac{3n}{4n-1}\right)}_{\frac{3}{4}} = \text{DNE} \neq 0$.

By D.T.T. $\sum b_n$ is divergent.

c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{2^n} \overset{\text{let}}{=} a_n$.

i. $a_{n+1} = \dfrac{1}{2^{n+1}} \leq \dfrac{1}{2^n} = a_n$ ✓

ii. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$ ✓

$\therefore$ by ALT. $\Rightarrow \displaystyle\sum (-1)^n \cdot \frac{1}{2^n}$ is convergent.

d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$; Let $a_n = \dfrac{n^2}{n^3+1}$; How do we know $a_n = \dfrac{n^2}{n^3+1}$ is decreasing, consider the following function $f(x) = \dfrac{x^2}{x^3+1} \Rightarrow f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2} \Rightarrow f'(x) < 0$ for $x > \sqrt[3]{2}$ i.e.

$a_n = \dfrac{n^2}{n^3+1}$ is decreasing.

$\displaystyle\lim_{n\to\infty}\frac{n^2}{n^3+1} = 0$. By the Alternating Series Test, $a_n = \dfrac{n^2}{n^3+1}$ is convergent.

e) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{3n+2}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{3n+2}} = \sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{\sqrt{3n+2}} \stackrel{\text{let}}{=\!=} a_n.$

i. $\displaystyle a_{n+1} = \frac{1}{\sqrt{3(n+1)+2}} \leq \frac{1}{\sqrt{3n+2}} = a_n$

ii. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{3n+2}} = 0$

$\therefore$ By ALT: $\sum(-1)^n\cdot a_n$ is convergent.

f) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{n^4+5}$ ✓

Note: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ → Divergent by P-Test (Harmonic Series)

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ → Convergent by ALT. (Alt. Harmonic Series)

Error Analysis.

Estimating Sums:

Alternating Series Estimation Theorem: If $S=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is the sum of an alternating series that satisfies

i) $0 \le a_{n+1} \le a_n$ and ii) $\lim_{n\to\infty} a_n = 0$ Then $|R_n| = |S - S_n| \le |a_{n+1}|$

$\text{Error} = |R_n| = |S - S_n| \leq |a_{n+1}|$

"the next term"

$\text{Error} = |R_{10}| \leq |a_{11}|$

Ex: Approximate the sum of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ with an error of less than 0.01.

$n = ?$ so that error $= |R_n| < 0.01$.

$|R_n| \le |a_{n+1}|$ where $a_n = \frac{1}{n}$; $a_{n+1} = \frac{1}{n+1}$

$|R_n| \le \frac{1}{n+1} \le 0.01$

$\frac{1}{0.01} \le n+1$

$100 \le n+1$

$\boxed{n \ge 99}$

Ex: How many terms are needed in computing the sum of $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2n+4}}$ to ensure its accuracy to 0.001

Sol: for ALT: Error $= |R_n| < |a_{n+1}|$

where $a_n = \frac{1}{n^3+2n+4}$.

Error $= |R_n| \le |a_{n+1}| = \frac{1}{(n+1)^3+2(n+1)+4} \le 0.001$

Trial & error: $n=4 \Rightarrow \frac{1}{5^3+2(5)+4} = \frac{1}{139} = 0.007 > 0.001$

$n = 8 \Rightarrow \frac{1}{9^3+2(9)+4} = 0.00133 > 0.001$

$n = 9 = \frac{1}{10^3+20+4} = \frac{1}{1024} = 0.0009 < 0.001$

Error:

$$\begin{cases} \text{For } \underline{\text{I.T.T.}} & \text{Error} = |R_n| \leq \int_n^{\infty} f(x)\,dx \\ \text{For ALT.} & \text{Error} = |R_n| \leq |a_{n+1}| \end{cases}$$

Ex: Find n so that $|R_n| < 0.0001$

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{[5n^2+1]^4}$ ← I.T.T. b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n^4+2}$ ← ALT.

Sol: a) $\text{Error} = |R_n| \leq \int_n^{\infty} f(x)\,dx$

where $f(x) = \dfrac{x}{[5x^2+1]^4}$

$$|R_n| \leq \int_n^{\infty} \frac{x}{[5x^2+1]^4}\,dx \qquad \begin{cases} \text{let } u = 5x^2+1 \\ \dfrac{du}{10} = x\,dx \end{cases}$$

$$= \lim_{t\to\infty} \int \frac{\frac{1}{10}du}{u^4} = \frac{1}{10} \lim_{t\to\infty} \int u^{-4}\,du.$$

$$= \frac{1}{10} \lim_{t\to\infty} \frac{u^{-3}}{-3} = -\frac{1}{30} \lim_{t\to\infty} \frac{1}{(5x^2+1)^3} \Bigg|_{n}^{t}.$$

$$= -\frac{1}{30} \lim_{t\to\infty} \left[\underbrace{\frac{1}{(5t^2+1)^3}}_{=0} - \frac{1}{(5n^2+1)^3} \right].$$

$$|R_n| \leq \frac{1}{30(5n^2+1)^3} < 0.0001$$

$$n=10 \rightarrow \frac{1}{30(501)^3} = 2.65 \times 10^{-10} < 0.0001$$

b) $$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n^4+2}$$

Error $= |R_n| \leq |a_{n+1}|$ where $a_n = \frac{n}{n^4+2}$.

$$|R_n| \leq |a_{n+1}| = \frac{n+1}{(n+1)^4+2} \leq 0.0001$$

$$n=10 \Rightarrow \frac{11}{11^4+2} = 0.00075 > 0.0001$$

$$n=20 \Rightarrow \frac{21}{21^4+2} = 0.000107 > 0.0001$$

$n = 25$