Section 7.2 ***Trigonometric Integrals***

For $\int \sin^{m} x \cos^{n} x\,dx \Rightarrow \begin{cases} \text{if } m \text{ is odd} \Rightarrow \text{let } u = \cos(x) \\ \text{if } n \text{ is odd} \Rightarrow \text{let } u = \sin(x) \end{cases}$ positive.

$(A)^{odd} \Rightarrow$ let $u = B$; $B^{odd} \Rightarrow$ let $u = A$ } let u be "the other funct".

Ex: Integrate the following:

a) $\int \sin^4(3x)\cos^5(3x)\,dx = \int \sin^4(3x)\cdot\cos^4(3x)\cdot\cos(3x)\,dx$

Let $\boxed{u = \sin(3x)}$

$du = 3\cos(3x)\,dx$

$\frac{du}{3} = \cos(3x)\,dx$

$= \int \sin^4(3x)\cdot\left(1-\sin^2(3x)\right)^2\cdot\cos(3x)\,dx$

$= \int u^4\left(1-u^2\right)^2\frac{du}{3} = \frac{1}{3}\int u^4\left(1-2u^2+u^4\right)du.$

$= \frac{1}{3}\int\left(u^4-2u^6+u^8\right)du = \frac{1}{3}\left[\frac{1}{5}u^5-\frac{2}{7}u^7+\frac{1}{9}u^9\right]+C$

$= \frac{1}{3}\left[\frac{1}{5}\sin^5(3x)-\frac{2}{7}\sin^7(3x)+\frac{1}{9}\sin^9(3x)\right]+C.$

b) $\int \sqrt[5]{\cos^3(2x)}\,\sin^3(2x)\,dx = \int\sqrt[5]{\cos^3(2x)}\cdot\sin^2(2x)\cdot\sin(2x)\,dx.$

Let $u = \cos(2x)$

$du = -2\sin(2x)\,dx$

$\frac{du}{-2} = \sin(2x)\,dx$

$= \int\sqrt[5]{\cos^3(2x)}\cdot\left(1-\cos^2(2x)\right)\cdot\sin(2x)\,dx$

$= \int\sqrt[5]{u^3}\left(1-u^2\right)\cdot\frac{du}{-2} = -\frac{1}{2}\int\left(u^{3/5}-u^{13/5}\right)du.$

$= -\frac{1}{2}\left[\frac{5}{8}u^{8/5}-\frac{5}{18}u^{18/5}\right]+C = -\frac{1}{2}\left[\frac{5}{8}\left(\cos(2x)\right)^{8/5}-\frac{5}{18}\left(\cos(2x)\right)^{18/5}\right]+C.$

b) $\int\frac{\cos^5(3x)}{\sqrt[4]{\sin^3(3x)}}dx = \int\frac{\cos^4(3x)}{\sqrt[4]{\sin^3(3x)}}\cdot\cos(3x)\,dx = \int\frac{\left(1-\sin^2(3x)\right)^2}{\sqrt[4]{\sin^3(3x)}}\cdot\cos(3x)\,dx.$

Let $u = \sin(3x)$ ✓

$du = 3\cos(3x)\,dx$

$\frac{du}{3} = \boxed{\cos(3x)\,dx}$

$= \int\frac{\left(1-u^2\right)^2}{u^{3/4}}\cdot\frac{du}{3} = \frac{1}{3}\int\frac{1-2u^2+u^4}{u^{3/4}}du.$

$= \frac{1}{3}\int\left(u^{-3/4}-2u^{5/4}+u^{13/4}\right)du = \frac{1}{3}\left[4\left(\sin(3x)\right)^{1/4}-2\cdot\frac{4}{9}\left(\sin(3x)\right)^{9/4}+\frac{4}{17}\left(\sin(3x)\right)^{17/4}\right]+C$

What if none of the power is odd ➔ $\boxed{\text{Double angle formulas}}$

$\to \cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$

$\to \sin(\theta+\beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$.

$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta \Rightarrow \boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$

$\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1 \quad \boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$

$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta \quad \boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$

Know these.

c) $\int \cos^4(3x)\,dx = \int (\cos^2(3x))^2 dx = \int \left(\frac{1+\cos(6x)}{2}\right)^2 dx = \frac{1}{4}\int \left[1 + 2\cos(6x) + \cos^2(6x)\right] dx$

$= 3/2$

$= \frac{1}{4}\int \left[1 + 2\cos(6x) + \frac{1+\cos(12x)}{2}\right] dx$

$= \frac{1}{4}\int \left[\frac{3}{2} + 2\cos(6x) + \frac{1}{2}\cos(12x)\right] dx$

$= \frac{1}{4}\left[\frac{3}{2}x + \frac{1}{3}\sin(6x) + \frac{1}{24}\sin(12x)\right] + C$

$\boxed{\text{Try } \int \sin^6(2x)\,dx = \ldots}$

For tangent / cotangent / secant / cosecant functions

Ex: Evaluate

$\int \tan x\,dx = \ln|\sec x| + C$

$\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \csc x\,dx = \ln|\csc x - \cot x| + C$

$\int \cot x\,dx = \ln|\sin x| + C$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

$(\text{secant}^{\text{odd}}) \to$ let $u = \text{secant}$

Ex: Integrate the following:

$(\text{secant})^{\text{even}} \rightarrow$ let $u =$ tangent.

If it does not work

$\Rightarrow$ Convert to sine & cosine.

$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

a) $\int \sec^3(4x)\tan^3(4x)\,dx$

let $u = \sec(4x)$

$du = 4\sec(4x)\tan(4x)dx$

$\frac{du}{4} = \sec(4x)\tan(4x)dx$

$\int \sec^2(4x)\cdot\tan^2(4x)\cdot\sec(4x)\tan(4x)dx$

$= \int \sec^2(4x)\left(\sec^2(4x)-1\right)\cdot\sec(4x)\tan(4x)\,dx$

$= \int u^2(u^2-1)\frac{du}{4} = \frac{1}{4}\int (u^4 - u^2)\,du.$

$= \frac{1}{4}\left[\frac{1}{5}\sec^5(4x) - \frac{1}{3}\sec^3(4x)\right] + C$

b) $\int \sec^4(3x)\tan^4(3x)\,dx = \int \sec^2(3x)\cdot\underbrace{\tan^4(3x)}_{u^4}\cdot\underbrace{\sec^2(3x)dx}_{\frac{du}{3}}$

Let $u = \tan(3x)$

$du = 3\sec^2(3x)dx.$

$\frac{du}{3} = \sec^2(3x)dx$

$= \int \left(1+\tan^2(3x)\right)\cdot\tan^4(3x)\cdot\sec^2(3x)dx$

$= \int (1+u^2)\,u^4\cdot\frac{du}{3}$

$= \frac{1}{3}\int (u^4 + u^6)\,du = \frac{1}{3}\left[\frac{1}{5}\tan^5(3x) + \frac{1}{7}\tan^7(3x)\right] + C$

c) $\int \sec^4(2x)\tan^3(2x)\,dx = \int \sec^2(2x)\cdot\tan^3(2x)\cdot\sec^2(2x)dx$

let $u = \tan(2x)$

$du = 2\sec^2(2x)dx$

$\frac{du}{2} = \sec^2(2x)dx$

$= \int \left(1+\tan^2(2x)\right)\cdot\tan^3(2x)\cdot\sec^2(2x)dx$

$= \int (1+u^2)\cdot u^3\cdot\frac{du}{2}$

$= \frac{1}{2}\int (u^3 + u^5)\,du = \frac{1}{2}\left[\frac{1}{4}\tan^4(2x) + \frac{1}{6}\tan^6(2x)\right] + C.$

d) $\int \tan^3(4x)\,dx = \int \tan^2(4x)\cdot\tan(4x)\,dx = \int (\sec^2(4x)-1)\cdot\tan(4x)\,dx.$

$$\boxed{\int \frac{\sin^3(4x)}{\cos^3(4x)}dx \quad u=\cos(4x)}$$

$$= \underbrace{\int \sec^2(4x)\tan(4x)\,dx}_{} - \underbrace{\int \tan(4x)\,dx}_{}$$

$u=\tan(4x)$

$\frac{du}{4} = \sec^2(4x)\,dx$

$$= \frac{1}{4}\int u\,du - \frac{1}{4}\ln|\sec(4x)| + C.$$

$$= \frac{1}{8}\tan^2(4x) - \frac{1}{4}\ln|\sec(4x)| + C.$$

$\frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$

$\cot^2 x + 1 = \csc^2 x.$

d) $\int \csc^4(5x)\,dx = \int \csc^2(5x)\cdot\csc^2(5x)\,dx$

$$= \int (\cot^2(5x)+1)\cdot\csc^2(5x)\,dx$$

$u = \cot(5x)$

$du = -5\csc^2(5x)\,dx$

$\frac{du}{-5} = \csc^2(5x)\,dx$

$$= \int (u^2+1)\cdot\frac{du}{-5} = -\frac{1}{5}\left[\frac{1}{3}\cot^3(5x) + \cot(5x)\right] + C.$$

b) $\int \sec^3 x\,dx \rightarrow$ Reduction Formula:

$$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

$$\int \sec^3 x\,dx = \frac{\sec x\tan x}{2} + \frac{1}{2}\underbrace{\int \sec x\,dx}_{}.$$

$$= \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Evaluating $\int \sin(mx)\cos(nx)dx; \int \sin(mx)\sin(nx)dx; \int \cos mx \cos nx dx$

a) $\sin A \cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$

b) $\sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$

c) $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Ex: Evaluate:

a) $\int \sin(\underset{A}{7x})\cos(\underset{B}{5x})dx = \frac{1}{2}\int[\sin(7x-5x)+\sin(7x+5x)]dx$.

$$= \frac{1}{2}\int[\sin(2x)+\sin(12x)]dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos(2x)-\frac{1}{12}\cos(12x)\right]+C.$$

b) $\int \cos(\underset{A}{4x})\cos(\underset{B}{3x})dx = \frac{1}{2}\int[\cos(4x-3x)+\cos(4x+3x)]dx$

$$= \frac{1}{2}\int[\cos x+\cos(7x)]dx$$

$$= \frac{1}{2}\left[\sin x+\frac{1}{7}\sin(7x)\right]+C.$$