Section 11.6 Absolute convergence and The Ratio and Root Tests

Def: A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$\sum a_n$ is absolutely convergent when $\sum |a_n|$ is convergent.

Determine whether $\sum a_n$ is abs. convergent.

Ex: ~~The series:~~

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} = a_n$. $\Rightarrow$ Consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$

$= \sum \left(\frac{1}{3}\right)^n \left\{ \begin{array}{l} r = \left|\frac{1}{3}\right| < 1 \\ \text{is convergent by G.S.} \end{array} \right.$

$\Rightarrow$ By def. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ is abs. convergent

b) $\sum_{n=1}^{\infty} \frac{\sin(5n - \pi)}{n^3} = a_n$

Consider $\sum |a_n| = \sum_{n=1}^{\infty} \left| \frac{\sin(5n-\pi)}{n^3} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$. $\left\{ \begin{array}{l} p = 3 > 1 \text{ is} \\ \text{convergent} \\ \text{by P-Test} \end{array} \right.$ ✓

b/c $|\sin(5n-\pi)| \leq 1$

$\therefore$ By D.C.T.T. $\Rightarrow \sum |a_n|$ is also convergent

$\therefore$ By def. $\Rightarrow \sum a_n$ is abs. convergent.

Def: A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

i.e. $\sum a_n$ is convergent, $\sum |a_n|$ is divergent

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by ALT.

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-Test

$\Rightarrow \sum \frac{(-1)^n}{n}$ is conditionally convergent.

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{\sqrt{3n^7 + 2n^5 + 1}}$ let a_n.

i. $a_{n+1} = \frac{(n+1)^3}{\sqrt{3(n+1)^7 + 2(n+1)^5 + 1}} \leq \frac{n^3}{\sqrt{3n^7 + 2n^5 + 1}} = a_n$

ii. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{\sqrt{3n^7 + 2n^5 + 1}} = 0$

By A.L.T. $\Rightarrow \sum (-1)^n \cdot a_n$ is convergent.

Consider $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot n^3}{\sqrt{3n^7 + 2n + 1}} \right| = \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{3n^7 + 2n + 1}} \geq \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{3n^7 + 2n^7 + n^7}}$

$= \sum \frac{n^3}{\sqrt{6n^7}} = \frac{1}{\sqrt{6}} \sum \frac{1}{n^{\frac{7}{2} - 3}} = \frac{1}{\sqrt{6}} \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent by p-Test

$\Rightarrow \sum |a_n|$ is divergent

$\Rightarrow$ it's conditionally convergent.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

$\sum a_n$: abs. convergent if $\sum |a_n|$ is convergent

$\sum a_n$: conditional convergent if $\begin{cases} \sum a_n \text{ is convergent} \\ \sum |a_n| \text{ is divergent} \end{cases}$

Ex: Test for convergence / divergence:

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+2)}{n^5+2n+7} = a_n$.

Consider $\sum |a_n| = \sum_{n=1}^{\infty} \frac{n+2}{n^5+2n+7} \le \sum_{n=1}^{\infty} \frac{n+2n}{n^5} = \sum_{n=1}^{\infty} \frac{3n}{n^5} = 3\sum_{n=1}^{\infty} \frac{1}{n^4}$

$p = 4 > 1$ is convergent by p-Test

$\Rightarrow$ By D.C.T.T. $\sum |a_n|$ is convergent

By def. $\sum a_n$ is absolutely convergent

$\Rightarrow \sum a_n$ is convergent.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{\sqrt{7n^8+4n+1}} \stackrel{\text{let}}{=} a_n$.

$\sum |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot (n+2)}{\sqrt{7n^8+4n+1}} \right| = \sum_{n=1}^{\infty} \frac{n+2}{\sqrt{7n^8+4n+1}}$

$\le \sum \frac{n+2\cdot n}{\sqrt{n^8}} = 3\sum \frac{n}{n^4} = 3\sum \frac{1}{n^3}$

By D.C.T.T. $\Rightarrow \sum |a_n|$ is convergent

$\Rightarrow$ By def. $\sum a_n$ is abs. convergent.

b) $\sum_{n=1}^{\infty} (-1)^n \left[\ln(2n+1) - \ln(n+2) \right]$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \ln\left(\frac{2n+1}{n+2}\right) \stackrel{\text{let}}{=} a_n$

$\lim_{n\to\infty} a_n = \underbrace{\lim_{n\to\infty} (-1)^n}_{= \text{DNE}} \cdot \underbrace{\lim_{n\to\infty} \left(\ln\left(\frac{2n+1}{n+2}\right)\right)}_{\ln 2} = \text{DNE} \ne 0 \Rightarrow$ Divergent by D.T.T.

c) $\sum_{n=1}^{\infty} \frac{(-3)^n}{7^n+4^n} \stackrel{\text{let}}{=} a_n$

$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-3)^n}{7^n+4^n} \right| = \sum_{n=1}^{\infty} \frac{3^n}{7^n + 4^n} \le \sum_{n=1}^{\infty} \frac{3^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n$ $\begin{cases} r = \left|\frac{3}{7}\right| < 1 \\ \text{is convergent} \\ \text{G.S.} \end{cases}$

$\therefore$ by D.C.T.T. $\Rightarrow \sum |a_n|$ is also convergent

By def. $\sum a_n$ is absolutely convergent.

$\Rightarrow \sum a_n$ is convergent.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} |a_n|$ diverges.

The Ratio Test: Given a series $\sum_{n=1}^{\infty} a_n$ $\Rightarrow$ Consider $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1, \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, $\Rightarrow$ Ratio-Test fails $\Rightarrow$ Use different method.

Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$

Ex: Test for the convergence:

a) $\sum_{n=1}^{\infty} \frac{n}{3^n}$ WT a_n. a_{n+1}, $\frac{1}{a_n}$.

Consider $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3^{n+1}}\cdot\frac{3^n}{n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{3n}\right) \overset{?}{=} \frac{1}{3} < 1$

By Ratio-Test: $\sum a_n$ is convergent.

b) $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n^3}{3^n}$ WT a_n.

Consider $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n (n+1)^3}{3^{n+1}}\cdot\frac{3^n}{(-1)^{n-1}\cdot n^3}\right|$

$= \lim_{n\to\infty}\frac{(n+1)^3}{3n^3} = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3} < 1$

$\Rightarrow$ By Ratio-Test $\Rightarrow \sum a_n$ is convergent.

Note: 1. Ratio-Test works well with factorial expressions.

2. $(n+1)! = (n+1) \cdot n!$

c) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$, $\frac{n^n}{n!} = a_n$ ✓

Consider:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| = \lim_{n\to\infty} \left(\frac{(n+1)^n \cdot (n+1) \cdot n!}{(n+1) \cdot n! \cdot n^n} \right) = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e^1 = e > 1$$

By the Ratio-Test $\Rightarrow \sum a_n$ is divergent.

d) $$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} \leq \sum_{n=1}^{\infty} \frac{2^n + 5 \cdot 2^n}{3^n} = \sum_{n=1}^{\infty} \frac{6 \cdot 2^n}{3^n} = \sum_{n=1}^{\infty} 6 \left(\frac{2}{3} \right)^n$$

$\left\{ r = \left| \frac{2}{3} \right| < 1 \right.$ is convergent by G. S.

$\therefore$ by D. C. T. T. $\Rightarrow \sum \frac{2^n + 5}{3^n}$ is convergent.

$\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} = \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \overset{\text{let}}{=\!=} a_n.$

Consider

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)! \cdot (n!)^2}{(n+1)^2 (n!)^2 \cdot (2n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(n+1)^2 \cdot (2n)!} = \lim_{n\to\infty} \left[\frac{2(2n+1)}{n+1} \right]$$

$$= \lim_{n\to\infty} \frac{4n+2}{n+1} = 4 > 1$$

Note: $(2n+2)! = (2n+2)(2n+1)(2n)!$

By Ratio-Test $\Rightarrow \sum a_n$ is divergent.

g) $\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$ Try this.

h) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n+1)}{(2\cdot 4\cdot 6\cdots(2n))\left[4^{n+2}+3\right]}$ let a_n.

Consider $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdots(2n+1)(2(n+1)+1)}{2\cdot 4\cdot 6\cdots(2n)\,(2(n+1))\,[4^{n+3}+3]} \cdot \frac{2\cdot 4\cdot 6\cdots(2n)\,[4^{(n+2)}+3]}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}\right|$.

$$= \lim_{n\to\infty}\left|\frac{(2n+3)\cdot(4^{n+2}+3)}{(2n+2)\cdot(4^{n+3}+3)}\right|$$

$$= \left[\lim_{n\to\infty}\frac{2n+3}{2n+2}\right]\cdot\left[\lim_{x\to\infty}\frac{4^{n+2}+3}{4^{n+3}+3} \div \frac{4^{n+3}}{4^{n+3}}\right]$$

$$= 1 \cdot \lim_{n\to\infty}\frac{\frac{1}{4}+\frac{3}{4^{n+3}}}{1+\frac{3}{4^{n+3}}} = \frac{1}{4} < 1$$

By Ratio-Test $\sum a_n$ is convergent.

The Root Test: Given a series $\sum_{n=1}^{\infty} a_n$ let $L = \lim_{n\to\infty}\sqrt[n]{a_n}$ → where $a_n = (f(n))^n$

i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (hence it's convergence)

ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ex: Test the convergence /divergence of the following series:

a) $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n \overset{\text{let}}{=} a_n = (f(n))^n$

$L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3} < 1$

$\therefore$ by Root-Test $\Rightarrow$ $\sum a_n$ is convergent

b) $\sum_{n=1}^{\infty}\left(\frac{7n^2+5n+4}{3n^2+2n+4}\right)^n \overset{\text{let}}{=} a_n.$

Consider $L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{7n^2+5n+4}{3n^2+2n+4}\right)^n}$

$= \lim_{n\to\infty}\frac{7n^2+5n+4}{3n^2+2n+4} = \frac{7}{3} > 1$

$\therefore$ By Root-Test $\Rightarrow$ $\sum a_n$ is divergent.

$\sum\left(\frac{7n+3}{7n+4}\right)^n \to a_n$

$L = \lim_{n\to\infty}\sqrt[n]{\left(\frac{7n+3}{7n+4}\right)^n} = \lim_{n\to\infty}\frac{7n+3}{7n+4} = 1$

Root-Test fails.

$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{7n+3}{7n+4} \div \frac{7n}{7n}\right)^n$

$= \lim_{n\to\infty}\frac{\left(1+\frac{3}{7n}\right)^n}{\left(1+\frac{4}{7n}\right)^n} = \frac{e^{3/7}}{e^{4/7}} \neq 0$

$\sum a_n$ is divergent by D.T.T.

☆ b) $\sum_{n=1}^{\infty}\left(\frac{2n+3}{2n+5}\right)^{n^2} = a_n.$

Consider: $L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{2n+5}\right)^{n^2}}$

$= \lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right)^n \doteq \lim_{n\to\infty}\frac{\left(1+\frac{3}{2n}\right)^n}{\left(1+\frac{5}{2n}\right)^n} = \frac{e^{3/2}}{e^{5/2}} = \frac{1}{e} < 1$

$\therefore$ by Root-Test $\sum a_n$ is convergent.

For Series.

1. G.S.
2. D.T.T.
3. T.S.
4. I.T.T. ⟵ Error : $|R_n| \leq \int_n^{\infty} f(x)\,dx$
5. P-Test.
6. D.C.T.T.
7. L.C.T.T.
8. ALT. ⟵ Error : $|R_n| \leq |a_{n+1}|$
9. Abs. Convergent.
10. Ratio-Test
11. Root-Test