Section 11.8 Power Series

Def: A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

Note: $\sum_{n=0}^{\infty} c_n (x-0)^n$: power series centered at $x=0$

$\sum_{n=0}^{\infty} c_n (x-a)^n$: power series centered at $x=a$

General form: $\sum_{n=0}^{\infty} c_n (x-a)^n$: power series centered at a or a power series about a.

We know that $\sum_{n=0}^{\infty} x^n$ is convergence if and only if $|x| < 1$

Ex: Find the $\boxed{\text{interval of convergence}}$ of $\sum_{n=1}^{\infty} \underbrace{\frac{n}{2^n}(x-1)^n}_{= a_n}$ $\Rightarrow$ $x = ?$ such that the series is convergent

Use Ratio-Test:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x-1)^n} \right|$$

$(x-1)$ $\quad 2$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(x-1)}{2n} \right| = \left| \frac{x-1}{2} \right| \cdot \underbrace{\lim_{n\to\infty} \left(\frac{n+1}{n} \right)} = \left| \frac{x-1}{2} \right|$$

For convergence $\Rightarrow$ $L < 1 \Rightarrow \left| \frac{x-1}{2} \right| < 1 \Rightarrow -1 < \frac{x-1}{2} < 1$

$-2 < x-1 < 2$

$\boxed{-1 < x < 3}$ ← Ans.

Ex: Find interval of convergence:

a) $\sum_{n=0}^{\infty} n!x^n \underline{\underline{\text{let}}}\ a_n$

$-1 \quad 1 \quad 3$

$\underbrace{\quad}_{2}\ \underbrace{\quad}_{2}$ ← Radius of convergence

(a, b)

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(n+1)!}^{\,n+1} \cdot \cancel{x^{n+1}}^{\,x}}{\cancel{n!} \cdot \cancel{x^n}} \right|$$

$$L = \lim_{n\to\infty} |(n+1)x|$$

$$= |x| \lim_{n\to\infty} (n+1) = \infty$$

For convergence $\Rightarrow$ $x = 0$

Interval of convergence: $\{0\}$

b) $\sum_{n=0}^{\infty} \frac{x^n}{n!} \underline{\underline{\text{let}}}\ a_n.$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}^{\,x}}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right|.$$

$$= |x| \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \text{ for } \underline{\text{all x-values}},$$

Interval of convergence: $(-\infty, \infty)$

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

i) The series converges only when x = a.

ii) The series converges for all x.

iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

Ex: Determine interval of convergence:

a) $\sum_{n=1}^{\infty} \frac{n}{2^n}(x-1)^n$

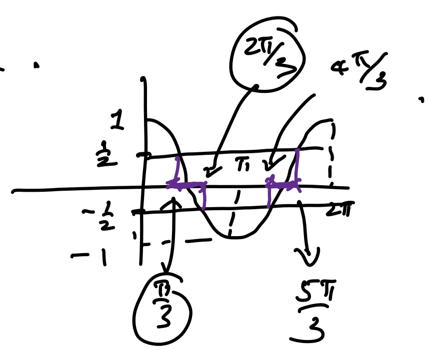
b) $\sum_{n=0}^{\infty} 2^n \cos^n x \overset{\text{let}}{=} a_n$.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{2^{n+1}} \cdot \cancel{\cos^{n+1} x}}{\cancel{2^n} \cdot \cancel{\cos^n x}} \right| = \lim_{n\to\infty} |2\cos x|$$

$$L = |2\cos x| < 1$$

For convergence.

$$-1 < 2\cos x < 1$$
$$-\frac{1}{2} < \cos x < \frac{1}{2}$$



$$\cos x = \frac{1}{2} \Rightarrow x = \begin{cases} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{cases}$$

$$\cos x = -\frac{1}{2} \Rightarrow x = \begin{cases} \frac{2\pi}{3} \\ \frac{4\pi}{3} \end{cases}$$

$$\boxed{\begin{array}{l} 2n\pi + \frac{\pi}{3} < x < \frac{2\pi}{3} + 2n\pi \\ 2n\pi + \frac{4\pi}{3} < x < \frac{5\pi}{3} + 2n\pi \end{array}}$$

c) $J_0 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ $\underline{\underline{\text{let}}}$ a_n, $\cancel{x^{2n+2}}$ $\boxed{x^2}$

$(-1) \Rightarrow = 1$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot x^{2(n+1)}}{2^{2(n+1)} \left((n+1)!\right)^2} \cdot \frac{\cancel{2^{2n}} \cdot \cancel{(n!)^2}}{(-1)^n \cdot \cancel{x^{2n}}} \right|$$

$\cancel{2^{2n+2}}$ 2^2 $\quad (n+1)^2 \cancel{(n!)^2}$

$$= \lim_{n\to\infty} \left| \frac{x^2}{4 \cdot (n+1)^2} \right| = \left| \frac{x^2}{4} \right| \lim_{n\to\infty} \left(\frac{1}{(n+1)^2} \right) = 0 < 1 \text{ for all } x\text{-values}$$

Interval of convergence: $(-\infty, \infty)$.

d) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ $\underline{\underline{\text{let}}}$ a_n.

$= (-3)\, x$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} \cancel{x^{n+1}}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{(-3)^n} \cdot \cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{-3x\sqrt{n+1}}{\sqrt{n+2}} \right|.$$

$$= |-3x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = |3x| < 1.$$

$= 1$

$-1 < 3x < 1$

$$\boxed{-\frac{1}{3} < x < \frac{1}{3}}$$

e) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ $\underline{\underline{\text{let}}}$ a_n.

$(x+2)$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{(x+2)^{n+1}}}{\cancel{3^{n+2}}} \cdot \frac{\cancel{3^{n+1}}}{n\cancel{(x+2)^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x+2)}{3n} \right|.$$

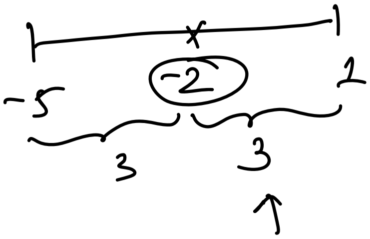
$= 3$

$$= \left| \frac{x+2}{3} \right| \cdot \underbrace{\lim_{n\to\infty} \left(\frac{n+1}{n} \right)}_{=1} = \left| \frac{x+2}{3} \right| < 1$$

$-1 < \frac{x+2}{3} < 1$

$-3 < x+2 < 3$

$\boxed{-5 < x < 1}$



Section 12.9 Representations of Functions as Power Series

$$\sum_{n=1}^{\infty} a x^{n-1} = \frac{a}{1-x} \text{ for } |x|<1$$

$$\sum_{n=0}^{\infty} a\cdot r^n = \frac{a}{1-r} \text{ if } |r|<1$$

let $\left.\begin{array}{l} a=1 \\ r=x \end{array}\right\} \Rightarrow \sum_{n=0}^{\infty} 1\cdot x^n = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = f(x)}$ for $|x|<1$

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x} \Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x|<1$$

General Case: $\boxed{f(x) = \frac{1}{1-A(x)} = \sum_{n=0}^{\infty} (A(x))^n \quad \text{for} \quad |A(x)|<1}$

This power series represents $f(x) = \frac{1}{1-A(x)}$.

Ex: Express the following expression as the sum of a power series and find the interval of convergence.

a) $f(x) = \frac{1}{1+2x^2} = \frac{1}{1-A(x)}$

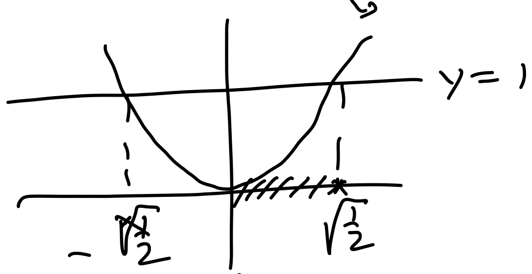
$$f(x) = \frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \cdot x^{2n}$$

Interval of convergence: $|A(x)|<1 \Rightarrow |-2x^2|<1$

$$-1 < 2x^2 < 1 \Rightarrow -\frac{1}{2} < x^2 < \frac{1}{2}$$

$$\boxed{-\sqrt{\tfrac{1}{2}} < x < \sqrt{\tfrac{1}{2}}}$$

$y = 2x^2$



$$f(x) = \frac{1}{1-A(x)} = \sum (Ax)^n$$

b) $\boxed{f(x)=\dfrac{9x-7}{x^2-x-6} \text{ about the } \boxed{\text{center } x=8}}$

$$f(x)=\frac{x^2}{5x^3+4}=\frac{1}{1-A(x)}=\sum_{n=0}^{\infty}(A(x))^n \qquad |A(x)|<1$$

minus

$$f(x)=\frac{x^2}{4}\cdot\frac{1}{1-\left(-\frac{5x^3}{4}\right)}=\frac{x^2}{4}\sum_{n=0}^{\infty}\left(-\frac{5x^3}{4}\right)^n$$

$A(x)$

$$=\frac{x^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 5^n\cdot x^{3n}}{4^n}=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 5^n\cdot x^{3n+2}}{4^{n+1}}$$

Interval of convergence: $|A(x)|<1 \Rightarrow \left|-\frac{5x^3}{4}\right|<1 \Rightarrow -1<\frac{5x^3}{4}<1$

$-4<5x^3<4$

$-\frac{4}{5}<x^3<\frac{4}{5}$

$=f(x)$

$\boxed{-\sqrt[3]{\frac{4}{5}}<x<\sqrt[3]{\frac{4}{5}}}$

Theorem: If the power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\ldots=\sum_{n=0}^{\infty}c_n(x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R,a+R)$ and

i) $f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+\ldots=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}$; $f''(x)=\sum_{n=2}^{\infty}n(n-1)c_n(x-a)^{n-2}$

ii) $\int f(x)dx=C+c_0(x-a)+c_1\frac{(x-a)^2}{2}+c_2\frac{(x-a)^3}{3}+\ldots=C+\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}$

$\int f(x)=\int c_0dx+c_1\int(x-a)dx+c_2\int(x-a)^2+\cdots$

$\int f(x)dx=c_0x+c_1\frac{(x-a)^2}{2}+\frac{c_2}{3}(x-a)^3+\cdots+C$

$= C_0 (x-a) \Rightarrow \sum_{n=0} C_n \frac{(x-a)}{n+1} + C.$

Ex: Find a power series of representation and interval of convergence

a) $f(x)=\ln(1+4x^7) = \int \frac{28x^6}{1+4x^7} dx = \int \frac{1}{1-A}$

$$= 28 \int x^6 \cdot \frac{1}{1-(-4x^7)} dx = 28 \int x^6 \sum_{n=0}^{\infty} (-4x^7)^n dx$$

$$= 28 \int \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \cdot x^{7n+6} dx = 28 \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \int x^{7n+6} dx$$

$$= 4 \cdot \cancel{7} \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \cdot \frac{x^{7n+7}}{\underbrace{7n+7}_{\cancel{7}(n+1)}} + C = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^{n+1}}{n+1} \cdot x^{7n+7} + C$$

b) ~~$f(x)=\frac{1}{(1-5x^3)^2}$~~

To find the constant C ⇒ find $f(0)$

$$f(0) = \ln 1 = 0 + C \Rightarrow C = \ln 1 = 0.$$

Ans: $f(x) = \ln(1+4x^7) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^{n+1}}{n+1} \cdot x^{7n+7}$

IOC: $|A(x)| < 1 \Rightarrow |-4x^7| < 1 \Rightarrow -1 < 4x^7 < 1$

$$\Rightarrow -\frac{1}{4} < x^7 < \frac{1}{4} \Rightarrow \boxed{-\sqrt[7]{\frac{1}{4}} < x < \sqrt[7]{\frac{1}{4}}}$$

c) $f(x) = \tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx.$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx.$$

$$f(x)=\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n+1} + C.$$

IOC:

$|-x^2| < 1$

$-1 < x^2 < 1$

$\boxed{-1 < x < 1}$

To find $C \Rightarrow f(0) = \tan^{-1}(0) = 0 + C$

$C = \tan^{-1}(0) = 0.$

$$\boxed{f(x) = \tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n+1}}$$

Ex: Evaluate the following as a power series:

a) $\int \frac{x^{1.4}}{2+x^7}\,dx = \int \frac{x^{1.4}}{2} \cdot \frac{1}{1-\left(-\frac{x^7}{2}\right)}\,dx.$

$$= \frac{1}{2}\int x^{1.4} \sum_{n=0}^{\infty} \left(-\frac{x^7}{2}\right)^n dx$$

$$= \frac{1}{2}\int x^{1.4} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{7n}}{2^n}\,dx.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \int x^{7n+1.4}\,dx.$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

$$\int \frac{x^{1.4}}{2+x^7}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \cdot \frac{x^{7n+2.4}}{7n+2.4} + C$$

No need to find C,

Know: $f(x) = \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot x^{2n+1}$

General Case:

$$\boxed{f(x) = \tan^{-1}(A(x)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (A(x))^{2n+1}}$$

Ex: Evaluate:

a) $\int \tan^{-1}(5x^7)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot (5x^7)^{2n+1} \cdot dx$

$(x^7)^{2n+1} = x^{14n+7}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot 5^{2n+1} \cdot \underbrace{\int x^{14n+7}\,dx}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot 5^{2n+1} \cdot \frac{x^{14n+8}}{14n+8} + C$$

$F(x)$