Section 11.10 ***Taylor and Maclaurin Series***

We have the power series as $\sum_{n=0}^{\infty} c_n (x-a)^n$. If we define a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Given any function: $f(x) \overset{\text{x/a at}}{=} \sum_{n=0}^{\infty} c_n (x-a)^n$, $x=a$ is a center,

$\Rightarrow$ Need to find c_n for $n = 0, 1, 2, \dots$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

$f(a) = c_0 \Rightarrow c_0 = f(a) = \frac{f(a)}{0!}$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$

$f'(a) = c_1 \Rightarrow c_1 = f'(a) = \frac{f'(a)}{1!}$

$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots$

$f''(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$

$f'''(x) = 2\cdot 3\cdot c_3 + 2\cdot 3\cdot 4c_4(x-a)^1 + 3\cdot 4\cdot 5c_5(x-a)^2$

$f'''(a) = 2\cdot 3\cdot c_3 \Rightarrow c_3 = \frac{f'''(a)}{2\cdot 3} = \frac{f'''(a)}{3!}$ $\dots$ $n=n \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$

Def: Let $f(x)$ has a power representation (expansion) at $x=a$. Where

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *where* $c_n = \frac{f^{(n)}(a)}{n!}$ is called a Taylor expansion of f(x) at $x=a$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Note: The Taylor polynomials of degree n at center $x=a$ $T_n(x,c) = \sum_{i=0}^{n} c_i (x-a)^i$

Def: Taylor expansion of f(x) at $x=0$ is called Maclaurin Series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Maclaurin Series of $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Ex: Find the Taylor series of the following function at the center x = a.

a) $f(x) = \frac{1}{x}$ at $a = 2$.

Taylor Series of $f(x)$ at $x = a$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Calculate $f^{(n)}(a) = ??$ for $n = 0, 1, \ldots$

Maclaurin Series of $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Calculate $f^{(n)}(0) = ??$ $n = 0, 1, \ldots$

$f(x) = \cos x$, $a = \frac{2\pi}{3}$

b) $f(x) = \sin(x)$ at $a = \frac{\pi}{3}$

$$f(x) = \sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{3})}{n!}\left(x - \frac{\pi}{3}\right)^n$$

Need to calculate $f^{(n)}(\frac{\pi}{3}) = ??$ for $n = 0, 1, 2, \ldots$

$n = 0 \Rightarrow f(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$n = 1 \Rightarrow f'(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$.

$n = 2 \Rightarrow f''(\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$.

$n = 3 \Rightarrow f'''(\frac{\pi}{3}) = -\cos(\frac{\pi}{3}) = -\frac{1}{2}$.

$n = 4 \Rightarrow f^{(4)}(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

$\frac{1}{2}$

6 $-\frac{\sqrt{3}}{2}$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \frac{\sqrt{3}}{2}}{(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \frac{1}{2}}{(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Ex: Find Maclaurin series of the following functions:

a) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ ⇒ Calculate: $f^{(n)}(0) = ??$ for $n = 0, 1, 2, \ldots$

$n=0 \Rightarrow f(0) = e^0 = 1$

$n=1 \Rightarrow f'(0) = e^0 = 1$

$n=2 \Rightarrow f''(0) = e^0 = 1$

$\vdots$

$n=n \Rightarrow f^{(n)}(0) = e^0 = 1$

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(Note: $\sum x^n = \frac{1}{1-x}$)

b) $f(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ Calculate $f^{(n)}(0) = ??$ for $n = 0, 1, 2, 3, \ldots$

$n=0 \Rightarrow f(0) = \cos(0) = 1$

$n=1 \Rightarrow f'(0) = -\sin(0) = 0$

$n=2 \Rightarrow f''(0) = -\cos(0) = -1$

$n=3 \Rightarrow f'''(0) = \sin(0) = 0$

$n=4 \Rightarrow f^{(4)}(0) = \cos(0) = 1$

6

$\vdots$

-1, 1, -1, $\vdots$

$$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

c) $f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ { Calculate $f^{(n)}(0) = ??$ for $n = 0, 1, 2, 3, \ldots$

$n=0 \Rightarrow f(0) = \sin(0) = 0$

$n=1 \Rightarrow f'(0) = \cos(0) = 1$

$n=2 \Rightarrow f''(0) = -\sin(0) = 0$

$n=3 \Rightarrow f'''(0) = -\cos(0) = -1$

$n=4 \Rightarrow f^{(4)}(0) = \sin(0) = 0$

5 — 1

7 — -1

1, -1

$\Rightarrow f(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$n=0$ $n=1$ $n=2$

Normally, we only interest at Taylor series up to certain degree n. So we $f(x) = T_n(x) + R_n(x)$, where $R_n(x)$ is the remainder (error) $R_n(x) = |f(x) - T_n(x)|$

Taylor's Theorem: If f is differentiable through order n + 1 in an open interval I containing a, then for each x in I, there exists a number c between x and a such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ where $|f^{(n+1)}(c)|$ is max.

Taylor's Inequality: If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$

Note: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real number x.

Must know:

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$

2. $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n+1}$

3. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$

5. $\sin x = \sum \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

} center $a = 0$.

$n=0 \quad (2n+1)!$

***Ex*:** Find the Maclaurin series of the function $f(x)=\sin x$. Show that this series converges to $\sin x$ for all real x.

$$f(x)=\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = a_n$$

$$\text{Ratio-Test } L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot x^{2(n+1)+1}}{(2(n+1)+1)!}\cdot\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x^2(2n+1)!}{(2n+3)!}\right|=|x^2|\lim_{n\to\infty}\left(\frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!}\right)$$

$$=|x^2|\cdot\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}=0 \text{ for all } x$$

$$IOC: \quad (-\infty,\infty)$$

***Ex*:** Find the Maclaurin series of the following functions:

a) $f(x)=x^3\cos(7x^2) = x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(7x^2)^{2n}=\sum_{n=0}\frac{(-1)^n\cdot 7^{2n}\cdot x^{4n+3}}{(2n)!}$

b) $f(x)=\dfrac{x^4}{e^{5x^3}} = x^4\cdot e^{-5x^3} = x^4\cdot\sum_{n=0}^{\infty}\frac{(-5x^3)^n}{n!}$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 5^n\cdot x^{3n+4}}{n!}$$

$$\sin x=\sum\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

c) $f(x)=\dfrac{\sin(3x^3)}{3x^2} = \dfrac{1}{3x^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n+1)!}(3x^3)^{2n+1}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot 3^{2n+1-1}\cdot x^{6n+3-2}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 3^{2n}\cdot x^{6n+1}}{(2n+1)!}$$

d) $f(x)=x^5\tan^{-1}(2x^3)$ $\left(\tan^{-1}x = \sum\dfrac{(-1)^n}{2n+1}\cdot x^{2n+1}\right).$

$$f(x)=x^5\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot(2x^3)^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n+1}\cdot x^{6n+3+5}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n+1}\cdot x^{6n+8}}{2n+1}$$

Note: $\sum \dfrac{}{(2n)!}$ ... → cosine

$\sum \dfrac{}{(2n+1)!}$ ... → $\sin x$

$\sum \dfrac{}{n!}$ ... → e^x.

$\sum \dfrac{}{2n+1}$ ... → $\tan^{-1}x$.

Ex: Using Maclaurin series to evaluate the following:

a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{\pi^{2n+1}}{2^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sin(x)$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1} = \sin\frac{\pi}{2} = 1.$$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\pi^{2n+1}}{9^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos(x)$ $\qquad \cos\frac{\pi}{3}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \pi^{2n} \cdot \pi}{(2n)!\ (3^2)^n} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \pi^{2n}}{(2n)!\ 3^{2n}} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \left(\frac{\pi}{3}\right)^{2n} = \pi \cos\frac{\pi}{3} = \boxed{\frac{\pi}{2}}$$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{2^n}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ $\qquad = e^{-2/3}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{n!\ 3^n \cdot 3^{-1}} = 3 \sum_{n=0}^{\infty} \frac{\left(-\frac{2}{3}\right)^n}{n!} = 3 \cdot e^{-2/3} = \boxed{\frac{3}{\sqrt[3]{e^2}}}$$

-3

d) $\sum_{n=0}^{\infty} \frac{(-1)^n (3)^{n/2}}{(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cdot x^{2n+1} = \tan^{-1}(x)$

$$3^{\frac{n}{2}} = x^{2n+1}$$

$$(3^{1/2})^n = (\sqrt{3})^n = (\sqrt{3})^{\frac{2n}{2}} = \left((\sqrt{3})^{1/2}\right)^{2n} = (\sqrt[4]{3})^{2n+1-1} = (\sqrt[4]{3})^{2n+1} \cdot (\sqrt[4]{3})^{-1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (\sqrt[4]{3})^{2n+1} \cdot (\sqrt[4]{3})^{-1}}{2n+1} = \frac{1}{\sqrt[4]{3}} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (\sqrt[4]{3})^{2n+1}}{2n+1}$$

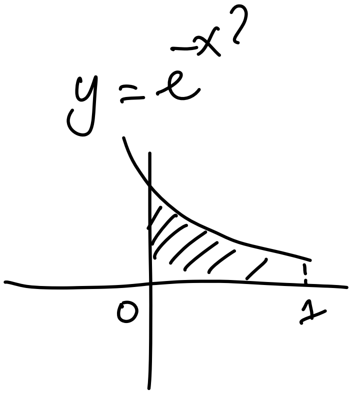
$$= \boxed{\frac{1}{\sqrt[4]{3}} \tan^{-1}(\sqrt[4]{3})}$$

Ex: a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series. $e^{x} = \sum \frac{x^n}{n!}$

Sol: $\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!} \cdot dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1}} + C$

b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.



$\int_0^1 e^{-x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1}\Big|_0^1$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}$ Let $= a_n$

For A.L.T. Error $= |R_n| \le |a_{n+1}| = \left|\frac{(-1)^{n+1}}{(n+1)!\,(2(n+1)+1)}\right|$

$\Rightarrow \left|\frac{1}{(n+1)!\,(2n+3)}\right| < 0.001$

Trial & error; $n=3 \Rightarrow \frac{1}{4!\,(9)} = \frac{1}{24(9)} = 0.0046 > 0.001$

$n=4 \Rightarrow \frac{1}{5!\,(11)} = \frac{1}{(120)(11)} = 0.0007 < 0.001$

$\int_0^1 e^{-x^2}dx \approx \sum^{4} \frac{(-1)^n}{\ldots} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = 0.75$

$$\int_0 e \, dx = \sum_{n=0} \frac{}{n!\,(2n+1)} \quad 3 \quad 10 \quad 42$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

Ex: Use power series to evaluate the following integrals:

a) $\int \frac{x^2}{3+5x^{15}}\,dx = \int \frac{x^2}{3}\cdot\frac{1}{1-\left(-\frac{5}{3}x^{15}\right)}\,dx = \frac{1}{3}\int x^2 \sum_{n=0}^{\infty}\left(-\frac{5}{3}x^{15}\right)^n dx.$

$\frac{1}{1-A(x)} = \sum (A(x))^n$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^n}{3^n} \int x^{15n+2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^n}{3^n}\cdot\frac{x^{15n+3}}{15n+3} + C$$

b) $\int x^2 \sin\left(x^{17}\right)dx = \int x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x^{17}\right)^{2n+1} dx$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{34n+17+2}\,dx.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{34n+20}}{34n+20} + C$$

c) $\int x^e e^{\sqrt{x}}\,dx = \int x^e \sum_{n=0}^{\infty} \frac{(\sqrt{x})^n}{n!}\,dx = \sum_{n=0}^{\infty} \frac{1}{n!}\int x^{\frac{n}{2}+e}\,dx.$

$e^x = \sum \frac{x^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{x^{\frac{n}{2}+e+1}}{\frac{n}{2}+e+1} + C$$

Multiplication and Division of Power Series:

Ex: Find the first three nonzero terms in the Maclaurin series for

a) $f\left(x\right)=e^{x}\sin x$

b) $f\left(x\right)=\tan x$

Note: A famous Euler's formula (Euler identity)

Prove the Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$