Section 11.11 *The Binomial Series*

From algebra, how do we expand two – term – expression ➔ Pascal Triangle ➔ Binomial for positive integer exponent.

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Pascal's Δ.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\cdot 7!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7!}}{1\cdot 2\cdot 3\cdot \cancel{7!}} = \boxed{120}$$

One of ~~Newton's~~ Taylor's accomplishments was to extend the Binomial Theorem to the case in which k is no longer a positive integer. In this case for $(a+b)^k$ is no longer a finite sum; it becomes an infinite series. Let's exam the Maclaurin series of $(1+x)^k$.

$$f(x) = (1+x)^k = \sum c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

$$n=0 \Rightarrow f(0) = (1+0)^k = 1$$
$$n=1 \Rightarrow f'(0) = k(1+0)^{k-1} = k$$
$$n=2 \Rightarrow f''(0) = k(k-1)(1+0)^{k-2} = k(k-1)$$
$$n=3 \Rightarrow f'''(0) = k(k-1)(k-2)(1+0)^{k-3} = k(k-1)(k-2)$$
$$n=4 \Rightarrow f^{(4)}(0) = k(k-1)(k-2)(k-3)$$
$$n=n \Rightarrow f^{(n)}(0) = k(k-1)(k-2)(k-3)\cdots(k-(n-1))$$

$$= (k(k-1)(k-2)(k-3)\cdots(k-n+1))$$

The Binomial Series: If k is any real number and $|x|<1$, then $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$ where

$k-(n-1)$
$k-n+1$

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \boxed{\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}} \text{ and } \binom{k}{0}=1$$

$(k-0)(k-1)(\cdots)(k-n+1)$

Binomial Series: $(1+x)^m = 1+\sum_{k=1}^{\infty}\binom{m}{k}x^k$ where we define $\binom{m}{1}=m$; $\binom{m}{2}=\frac{m(m-1)}{2!}$

And $\binom{m}{k} = \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}$ for $k \geq 3$

$$f(1+x)^k = 1+\sum_{n=1}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$$

$$= \binom{k}{n}$$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$$

Ex: Using Binomial series to expand: $f(x)=\frac{1}{(1+x)^2}$

$$\boxed{f(x) = (1+x)^k = \underbrace{1}_{n=0} + \sum_{n=1}^{\infty}\binom{k}{n}x^n \quad \text{for } k\in\mathbb{R}.}$$

Know this.

$$f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2} \quad \Big\{ \; k=-2.$$

$$f(x) = (1+x)^{-2} = \underbrace{1}_{n=0} + \sum_{n=1}^{\infty}\binom{-2}{n}\cdot x^n.$$

$n=1$: $\binom{-2}{1} = \frac{-2}{1!} = -2$

$n=2$: $\binom{-2}{2} = \frac{(-2)(-2-1)}{2!} = \frac{(-2)(-3)}{2} = +3$

$n=3$: $\binom{-2}{3} = \frac{(-2)(-2-1)(-2-2)}{3!} = \frac{(\cancel{-2})(\cancel{-3})(-4)}{1\cdot\cancel{2}\cdot\cancel{3}} = -4$

$n=4$: $\binom{-2}{4} = \frac{-2(-2-1)(-2-2)(-2-3)}{4!} = \frac{(\cancel{-2})(\cancel{-3})(\cancel{-4})(-5)}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}} = 5$

$4!$

b) $f(x) = \sqrt{1+x}$

$$f(x) = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \dots$$

$$f(x) = (1+x)^k = 1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n .$$

$n=0$

$k \in \mathbb{R}$

where $$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots\overbrace{(k-(n-1))}^{k-n+1}}{n!}$$

c) $f(x) = \sqrt[3]{1+x} = (1+x)^{1/3}$ $\Big\}$ $k = \frac{1}{3}$

$$f(x) = 1 + \sum_{n=1}^{\infty} \binom{1/3}{n} x^n$$

$n=1 \Rightarrow \binom{1/3}{1} = \dfrac{\frac{1}{3} - 0}{1!} = \dfrac{1}{3}$ ✓

$n=2 = \binom{1/3}{2} = \dfrac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!} = \dfrac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\cdot 2} = -\dfrac{1}{9}$ ✓

$n=3 \Rightarrow \binom{1/3}{3} = \dfrac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!} = \dfrac{\frac{1}{3}\left(-\frac{\cancel{2}}{3}\right)\left(-\frac{5}{3}\right)}{1\cdot \cancel{2}\cdot 3} = \dfrac{5}{81}$

$n=4 \Rightarrow \binom{1/3}{4} = \dfrac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!} = \dfrac{\frac{1}{3}\left(-\frac{\cancel{2}}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{\cancel{8}^{\,2}}{3}\right)}{1\cdot \cancel{2}\cdot 3\cdot \cancel{4}}$

$= -\dfrac{10}{3^5} = -\dfrac{10}{243}$

d) $f(x)=\frac{1}{(3+8x^3)^3}$

5 - terms

$f(x)=\sqrt[3]{1+x} = 1+\frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$

$n=0 \quad n=1 \quad n=2$

d) $f(x)=\frac{1}{(3+8x^3)^3} = (1+A(x))^k$

$=\frac{1}{3^3\left(1+\frac{8x^3}{3}\right)^3}=\frac{1}{27}\left(1+\frac{8x^3}{3}\right)^{-3=k}=\frac{1}{27}\left[1+\sum_{n=1}^{\infty}\binom{-3}{n}\left(\frac{8x^3}{3}\right)^n\right]$

$n=1 \Rightarrow \binom{-3}{1}=-3$

$n=2 \Rightarrow \binom{-3}{2}=\frac{-3(-3-1)}{2!}=\frac{12}{2}=6$

$n=3 \Rightarrow \binom{-3}{3}=\frac{-3(-3-1)(-3-2)}{3!}=\frac{-3(-4)(-5)}{1\cdot 2\cdot 3}=-10$

$n=4 \Rightarrow \binom{-3}{4}=\frac{-3(-3-1)(-3-2)(-3-3)}{4!}=\frac{-3(-4)(-5)(-6)}{1\cdot 2\cdot 3\cdot 4}=15$

$f(x)=\frac{1}{27}\left[1-3\left(\frac{8x^3}{3}\right)+6\left(\frac{8x^3}{3}\right)^2-10\left(\frac{8x^3}{3}\right)^3+15\left(\frac{8x^3}{3}\right)^4+\cdots\right]$

Section 11.12 Applications of Taylor Polynomials

<u>Approximating Functions by polynomials</u>

Suppose that $f(x)$ is equal to the sum of its Taylor series at a: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

So, we let $T_n(x)$be the first nth partial sum of this series and called it the nth-degree Taylor polynomial of f at a.

$$T_n(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i \text{ and let the error be } R_n(x)=\sum_{i=n+1}^{\infty}\frac{f^{(i)}(a)}{i!}(x-a)^i=|f(x)-T_n(x)|$$

We have from Taylor Inequality

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1} \text{ where } \left|f^{(n+1)}(x)\right|\le M$$

Ex: a) Approximate the function $f(x) = \sqrt[3]{x}$ by Taylor polynomial of degree 2 at $a = 8$

b) How accurate is this approximation when $7 \le x \le 9$?

Ex: The third Maclaurin polynomial for $\sin x$ is given by: $\sin x \approx x - \frac{x^3}{3!}$. Use Taylor's Theorem to approximate $\sin(0.1)$ by $T_3(0.1)$ and determine the accuracy of the approximation:

Ex: Determine the degree of the Taylor polynomial $T_n(x)$ expanded about $a = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.001.

Ex: Approximate $\sin 2^0$ accurate to four decimal places.

Ex: a) What is the maximum error possible in using the approximation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ where $-0.3 \le x \le 0.3$? Use this approximation to find $\sin 12^0$ corrects to six decimal places?

b) For what values of x is this approximation accurate to within 0.00005?