a: constant ; x: variable

let $u = \cdots$
$x = \cdots \theta$

$4 - x^2$
$x = 2\sin\theta$

Section 7.3 Trigonometric Substitution

know these.

1. $a^2 - x^2 \Rightarrow \boxed{x = a\sin\theta} \Rightarrow dx = a\cos\theta\, d\theta$
2. $\boxed{a^2 + x^2} \Rightarrow x = \boxed{a\tan\theta} \Rightarrow dx = a\sec^2\theta\, d\theta$
3. $x^2 - a^2 \Rightarrow x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

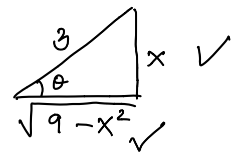
$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

Ex: Evaluate the following:

a) $\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{\sqrt{3^2 - x^2}}{x^2}dx$

let $x = 3\sin\theta \longrightarrow \sin\theta = \frac{x}{3} = \frac{\text{opp.}}{\text{Hyp.}}$

$dx = 3\cos\theta\, d\theta$



$\sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$.

$$= \int \frac{\sqrt{9 - 9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

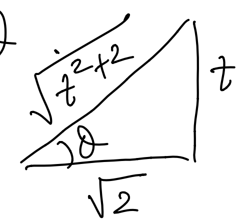
$$= \int \frac{\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta} \cdot \cancel{3}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \cot^2(\theta)\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

b) $\int \frac{t^5}{\sqrt{t^2+2}}dt = \int \frac{t^5}{\sqrt{t^2 + (\sqrt{2})^2}}dt$

let $t = \sqrt{2}\tan\theta \longrightarrow \tan\theta = \frac{t}{\sqrt{2}} = \frac{\text{opp.}}{\text{Adj}}$.

$dt = \sqrt{2}\sec^2\theta\, d\theta$



$$= \int \frac{(\sqrt{2}\tan\theta)^5}{\sqrt{2\tan^2\theta + 2}} \cdot \sqrt{2}\sec^2\theta\, d\theta =$$

$\sqrt{2(\tan^2\theta + 1)} = \sqrt{2\sec^2\theta} = \sqrt{2}\sec\theta$.

$$= \int \frac{\sqrt{32}\cdot\tan^5\theta}{\cancel{\sqrt{2}}\cdot\cancel{\sec\theta}} \cdot \cancel{\sqrt{2}}\sec^{\cancel{2}\,1}\theta\, d\theta = 4\sqrt{2}\int \tan^5\theta \cdot \sec\theta\, d\theta$$

$$= 4\sqrt{2}\int \tan^4\theta \cdot \underbrace{\tan\theta\sec\theta\, d\theta}_{= du}$$

let $u = \sec\theta \Rightarrow du = \sec\theta\tan\theta\, d\theta$

$$= 4\sqrt{2}\int (\sec^2\theta - 1)^2 \cdot \tan\theta\sec\theta\, d\theta = 4\sqrt{2}\int (u^2-1)^2 du = 4\sqrt{2}\int (u^4 - 2u^2 + 1)\, du.$$

$$= 4\sqrt{2}\left[\frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta\right] + C = 4\sqrt{2}\left[\frac{1}{5}\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^5 - \frac{2}{3}\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^3 + \frac{\sqrt{t^2+2}}{\sqrt{2}}\right] + C$$

c) $\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{dx}{x^2\sqrt{(4x)^2-3^2}}$

let $4x = 3\sec\theta \Rightarrow x = \frac{3}{4}\sec\theta$

$4dx = 3\sec\theta\tan\theta\, d\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\frac{3}{4}\sec\theta\tan\theta\, d\theta}{\frac{9}{16}\sec^2\theta\cdot\sqrt{9\sec^2\theta-9}}$$

$\sqrt{9(\sec^2\theta-1)} = \sqrt{9\tan^2\theta} = 3\tan\theta$.

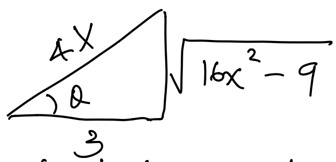
$$= \frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{16}^{\,4}}{\cancel{9}_{\,3}}\int \frac{\cancel{\sec\theta\tan\theta}\, d\theta}{\sec^{\cancel{2}}\theta\cdot 3\cancel{\tan\theta}} = \frac{4}{9}\int \frac{1}{\sec\theta}\, d\theta.$$

$$= \frac{4}{9}\int \cos\theta\, d\theta = \frac{4}{9}\sin\theta + C.$$

$$= \frac{4}{9}\cdot\frac{\sqrt{16x^2-9}}{4x} + C.$$

$4x = 3\sec\theta$

$\sec\theta = \frac{4x}{3} = \frac{\text{Hyp}}{\text{Adj.}}$



d) $\int \sqrt{5+4x-x^2}\, dx$

Note: If inside square root, there're 3 terms $\Rightarrow$ perform a completing square.

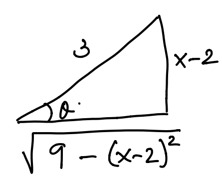
$5+4x-x^2 = 5-(x^2-4x+4)+4$

$= 9-(x^2-4x+4) = 9-(x-2)^2$

$= \int \sqrt{3^2-(x-2)^2}\, dx$

let $x-2 = 3\sin\theta \Rightarrow \sin\theta = \frac{x-2}{3} = \frac{\text{Opp.}}{\text{Hyp.}}$

$dx = 3\cos\theta\, d\theta$



$\sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$.

$$= \int \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

even $\to$ double angle formula,

$$= \int 3\cos\theta\cdot 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta = 9\int \frac{1+\cos(2\theta)}{2}\, d\theta.$$

$$= \frac{9}{2}\int [1+\cos(2\theta)]\, d\theta = \frac{9}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C,$$

$$= \frac{9}{2}\left[\theta + \frac{1}{\cancel{2}}\cdot\overbrace{\cancel{2}\sin\theta\cdot\cos\theta}^{\sin(2\theta)}\right] + C.$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\cdot\frac{\sqrt{9-(x-2)^2}}{3}\right] + C$$

$(a+b)^2 = a^2 + 2ab + b^2$

$$(a-b)^2 = a^2 - 2ab + b^2$$

c) $$\int \frac{3x-5}{\left(21+12x-9x^2\right)^{3/2}}\,dx$$

$$21+12x-9x^2 = 21 - \left[\underbrace{(3x)^2 - 2\cdot(3x)\cdot 2 + 4}_{(3x-2)^2}\right] + 4.$$

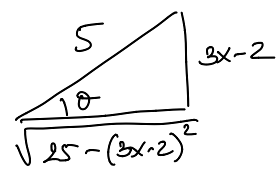
$$= 25 - (3x-2)^2.$$

$$\int \frac{(3x-2)-3}{\left[25-(3x-2)^2\right]^{3/2}}\,dx$$

let $3x-2 = 5\sin\theta \Rightarrow \sin\theta = \frac{3x-2}{5} = \frac{\text{opp.}}{\text{Hyp.}}$

$3dx = 5\cos\theta\, d\theta$

$dx = \frac{5}{3}\cos\theta\, d\theta$



$$= \int \frac{5\sin\theta - 3}{\left[25 - 25\sin^2\theta\right]^{3/2}} \cdot \frac{5}{3}\cos\theta\, d\theta = \int \frac{5\sin\theta - 3}{5^3 \cdot \cos^{\cancel{3}\,2}\theta} \cdot \frac{5}{3}\cos\theta\, d\theta = \frac{1}{75}\int \frac{5\sin\theta - 3}{\cos^2\theta}\,d\theta.$$

$$\left[25(1-\sin^2\theta)\right]^{3/2} = \left[\sqrt{25\cos^2\theta}\right]^3 = 5^3\cdot\cos^3\theta$$

$$= \frac{1}{75}\int \left[5\tan\theta\sec\theta - 3\sec^2\theta\right] d\theta$$

$$= \frac{1}{75}\left[5\sec\theta - 3\tan\theta\right] + C.$$

$$= \frac{1}{75}\left[5\cdot\frac{5}{\sqrt{25-(3x-2)^2}} - 3\cdot\frac{3x-2}{\sqrt{25-(3x-2)^2}}\right] + C$$

★ $$\int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dx}{\sqrt{(x-3)^2+2^2}}.$$

$x^2-6x+13 = x^2-6x+9+4$
$= (x-3)^2+4$

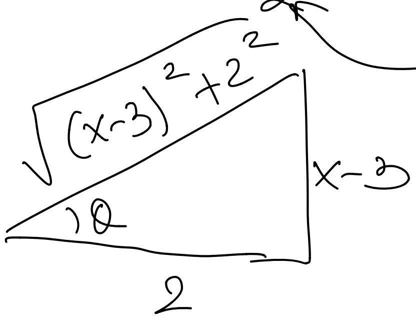
Let $x-3 = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$

$$= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\tan^2\theta + 4}}$$

$$\sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2\sec\theta.$$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{2\sec\theta}} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

$$\tan\theta = \frac{x-3}{2} = \frac{\text{opp.}}{\text{Adj.}} \qquad = \ln\left|\frac{\sqrt{(x-3)^2+4}}{2} + \frac{x-3}{2}\right| + C$$



g) $\displaystyle\int \frac{2x+4}{\left[-4x^2-12x-5\right]^{3/2}}\,dx$ Try this.

$$\int \frac{2x+3}{\left(4x^2-4x-3\right)^{5/2}}\,dx = \int \frac{(2x-1)+4}{\left[(2x-1)^2-4\right]^{5/2}}\,dx = \Rightarrow \text{Let } 2x-1 = 2\sec\theta$$

$$\cancel{2}dx = \cancel{2}\sec\theta\tan\theta\,d\theta$$

$$dx = \sec\theta\tan\theta\,d\theta$$

$$4x^2-4x-3 = (2x)^2 - 2(2x) + 1 - 1 - 3$$
$$a^2 - 2ab + b^2$$
$$= (2x-1)^2 - 4$$

$$= \int \frac{2\sec\theta + 4}{\left[4\sec^2\theta - 4\right]^{5/2}} \cdot \sec\theta\tan\theta\,d\theta.$$

$$\left[4\left(\sec^2\theta - 1\right)\right]^{5/2} = \left[4\cdot\tan^2\theta\right]^{5/2} = \left(2\cdot\tan\theta\right)^5 = 32\tan^5\theta.$$

$$= \int \frac{2\sec\theta + 4}{32\tan^{\cancel{5}}\theta} \cdot \sec\theta\cdot\cancel{\tan\theta}\,d\theta$$

$$= \frac{2}{32}\int \frac{\sec\theta + 2}{\tan^4\theta}\cdot\sec\theta\,d\theta = \frac{1}{16}\left[\underbrace{\int \frac{\sec^2\theta}{\tan^4\theta}\,d\theta}_{\substack{\text{let } u=\tan\theta \\ du=\sec^2\theta\,d\theta}} + 2\int \frac{\sec\theta}{\tan^4\theta}\,d\theta\right]$$

$$= \frac{1}{16}\left[\int \frac{du}{u^4} + 2\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cos^{\cancel{4}\,3}\theta}{\sin^4\theta}\,d\theta\right]$$

$$= \frac{1}{16}\left[\int u^{-4}\,du + 2\int \frac{\cos^3\theta}{\sin^4\theta}\,d\theta\right] \qquad \text{Let } u = \sin\theta,\quad du = \cos\theta\, d\theta.$$

$$= \frac{1}{16}\left[\frac{u^{-3}}{-3} + 2\int \frac{1-\sin^2\theta}{\sin^4\theta}\cdot\cos\theta\, d\theta\right].$$

$$= \frac{1}{16}\left[\frac{-1}{3\tan^3\theta} + 2\int\left(\frac{1-u^2}{u^4}\right)\cdot du\right].$$

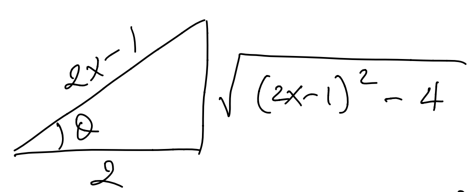
$$= \frac{1}{16}\left[-\frac{1}{3}\cot^3\theta + 2\int\left(u^{-4} - u^{-2}\right)du\right]$$

$$= \frac{1}{16}\left[-\frac{1}{3}\cot^3\theta + 2\left(\frac{\sin^{-3}(\theta)}{-3} + \frac{1}{\sin\theta}\right)\right] + C.$$

$$= \frac{1}{16}\left[-\frac{1}{3}\cot^3\theta - \frac{2}{3}\csc^3\theta + \csc\theta\right] + C.$$

from $2x-1 = 2\sec\theta$

$$\sec\theta = \frac{2x-1}{2} = \frac{\text{Hyp}}{\text{Adj}}.$$



$$= \frac{1}{16}\left[-\frac{1}{3}\left(\frac{2}{\sqrt{(2x-1)^2-4}}\right)^3 - \frac{2}{3}\left(\frac{2x-1}{\sqrt{(2x-1)^2-4}}\right)^3\right.$$

$$+ \frac{2x-1}{\sqrt{(2x-1)^2-4}} \Bigg] + C.$$