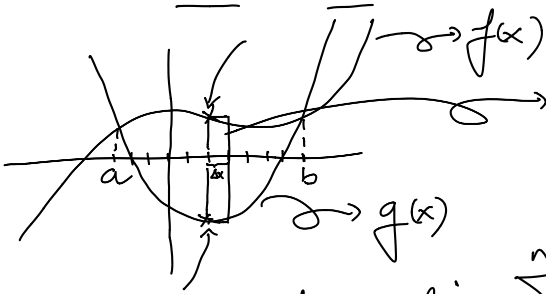
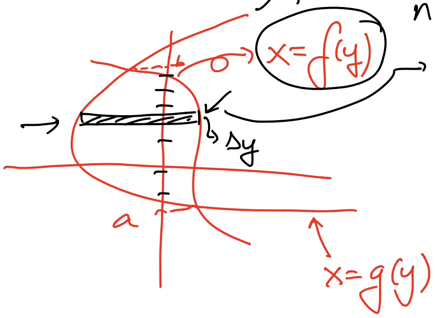
Section 6.1 ***Area between curves***

Given two functions $f(x)$ *and* $g(x)$ *over* $[a,b]$. How to find the area between the two curves:



$$\Delta x \quad y_{top} - y_{bot} = f(x) - g(x)$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(x_i) - g(x_i)\right)\Delta x = \int_a^b \left(f_{top}(x) - g_{bot}(x)\right) dx$$



$$\Delta y$$

$$L = x_{right} - x_{Left} = f(y) - g(y)$$

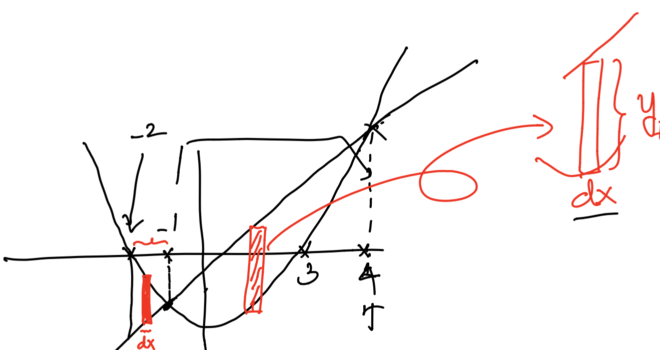
$$\boxed{\text{Area} = \int_a^b \left(f(y) - g(y)\right) dy}$$

Ex: Find the area bounded between

a) $y = x^2 - x - 6$ *and* $y = 2x - 2$ *for* ~~dx~~ $-2 \le x \le 4$

$= (x-3)(x+2) \Rightarrow x = -2, 3 \Rightarrow$ find pts of intersection $\Rightarrow$

$$x^2 - x - 6 = 2x - 2$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0 \Rightarrow x = 4, -1$$



$$y_{top} - y_{bot} = 2x - 2 - (x^2 - x - 6)$$
$$= 3x + 4 - x^2$$

dx

$$y_{top} - y_{bot} = x^2 - x - 6 - (2x - 2) = x^2 - 3x - 4$$

dx

$$\text{Area} = \int_{-2}^{-1} (x^2 - 3x - 4)\, dx + \int_{-1}^{4} (3x + 4 - x^2)\, dx.$$

Sketch & set up the integrals for area.

b) $x = -2y^2 - 3y + 2$ and $\cancel{x = 8}$ $x = 2y - 1$ for $-3 \le y \le 2$

$= -(2y^2 + 3y - 2)$

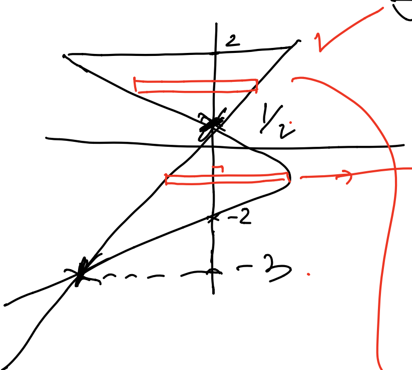
$= -(2y - 1)(y + 2)$

pts of intersection: $-2y^2 - 3y + 2 = 2y - 1$

$2y^2 + 5y - 3 = 0$

$(2y - 1)(y + 3) = 0$

$y = \frac{1}{2}, -3.$



dy

$x_{right} - x_{left} = -2y^2 - 3y + 2 - (2y - 1)$

$= -2y^2 - 5y + 3.$

$dy = 2y - 1 - (-2y^2 - 3y + 2) = 5y - 3 + 2y^2.$

$$\text{Area} = \int_{-3}^{1/2} (-2y^2 - 5y + 3)\,dy + \int_{1/2}^{2} (5y - 3 + 2y^2)\,dy.$$

c) $y = x^2 - x - 2$ and $y = -x^2 - x + 6$ for $-3 \le x \le 2$

$= (x - 2)(x + 1)$

$= -(x^2 + x - 6)$

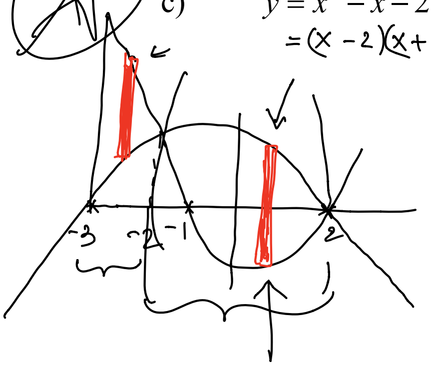
$= -(x + 3)(x - 2)$

pts of intersections:

$x^2 - x - 2 = -x^2 - x + 6$

$2x^2 - 8 = 0$

$x^2 - 4 = 0 \Rightarrow x = \pm 2.$



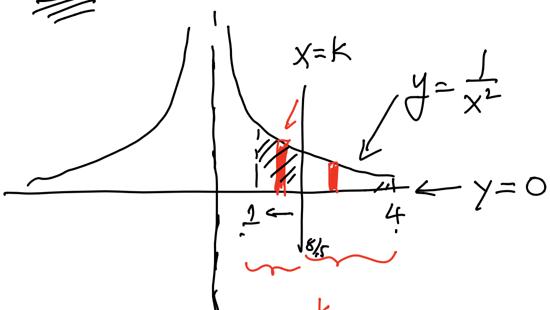
$$= \text{Area} = A = \int_{-3}^{-2} \left(x^2 - x - 2 - (-x^2 - x + 6)\right) dx + \int_{-2}^{2} \left[(-x^2 - x + 6) - (x^2 - x - 2)\right] dx$$

$$= \int_{-3}^{-2} (2x^2 - 8)\,dx + \int_{-2}^{2} (8 - 2x^2)\,dx,$$

$y = k \leftarrow$ Horizontal

$x = k \leftarrow$ Vertical

Ex. Find k so that $x = k$ bisects the area bounded by $y = \frac{1}{x^2}$ and $x - axis$ $1 \le x \le 4$



$$\text{Area} = \underbrace{\int_1^k \left(\frac{1}{x^2} - 0\right)dx}_{} = \underbrace{\int_k^4 \left(\frac{1}{x^2} - 0\right)dx}_{}$$

$$\int_1^k x^{-2}\,dx = \int_k^4 x^{-2}\,dx$$

$$\left.\frac{x^{-1}}{-1}\right|_1^k = \left.\frac{x^{-1}}{-1}\right|_k^4$$

$$\frac{1}{k} - 1 = \frac{1}{4} - \frac{1}{k}$$

$$\frac{1}{k} + \frac{1}{k} = \frac{1}{4} + 1$$

$$\frac{2}{k} = \frac{5}{4}$$

$$\boxed{k = \frac{8}{5}}$$

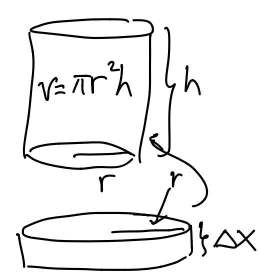
<u>Section 6.2</u> *Volumes (Dish method, Cross Section)*

From volume for basic shapes such as box, cylinder, sphere… We can find volume of irregular shape but uniform height, and then we find volume by the area of the base times the height.

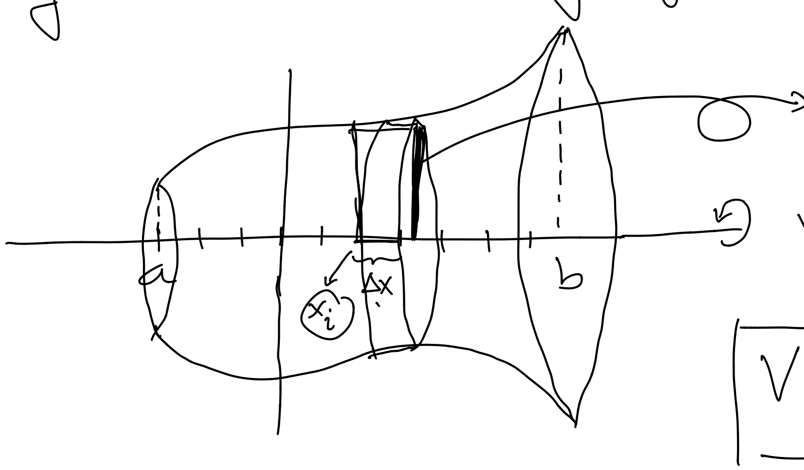
~~**Def**: Let S be a solid that lies between x = a and x = b. If the cross-sectional area of S in the plane P, through x and perpendicular to the x-axis, is A(x), where A is a continuous function, then the volume of S is~~

$$\cancel{V = \lim_{n\to\infty} \sum A(x_i^*)\Delta x = \int_a^b A(x)dx}$$

Volume of revolution: Volume of a cylinder



$$V = \pi r^2 \Delta x.$$
$$= \pi \cdot (f(x_i))^2 \cdot \Delta x$$

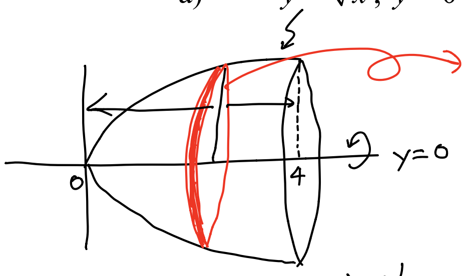


$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(f(x_i)\right)^2 \Delta x$$

$$\boxed{V = \pi \int_a^b \left(f(x)\right)^2 dx}$$

Ex: Find the volume of the following region bounded by

a) $y=\sqrt{x}$, $y=0$ and $x=4$, rotated about the x – axis.



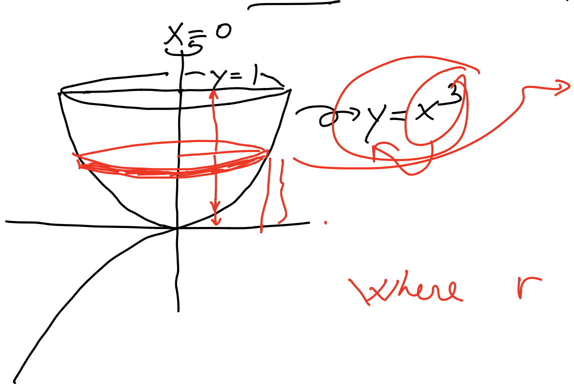
r
dx ← { If the disc is ⊥ to the x-axis, ⇒ dx
⊥ — y-axis ⇒ dy.

$V = \pi \cdot r^2 \cdot dx.$

where r : $y=\sqrt{x}$, $y=0$; $= y_{top} - y_{bot.} = \sqrt{x} - 0 = \sqrt{x}$

$$V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x\,dx = \frac{\pi}{2} \cdot x^2 \Big|_0^4 = \boxed{8\pi}$$

b) $y=x^3$ and $y-axis$ for $0 \le y \le 1$, rotated about the y-axis.



r
dy ← Solid disc.

$V = \pi \cdot r^2 \cdot dy$

where r : $x_r =$ $= x_{right} - x_{left}$

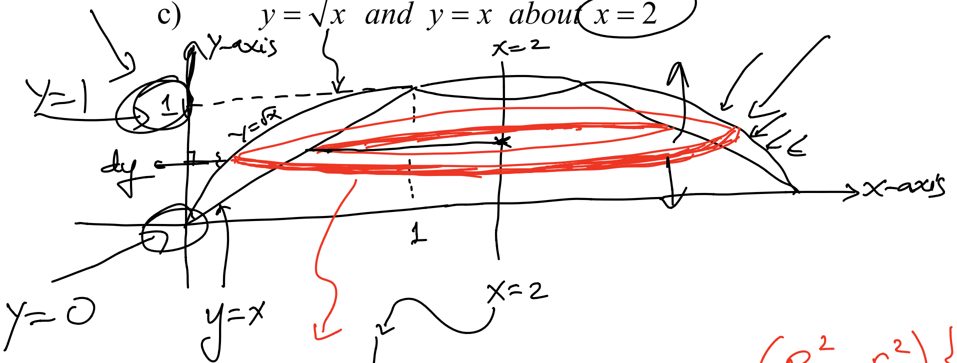
$= \sqrt[3]{y} - 0 = \sqrt[3]{y}$.

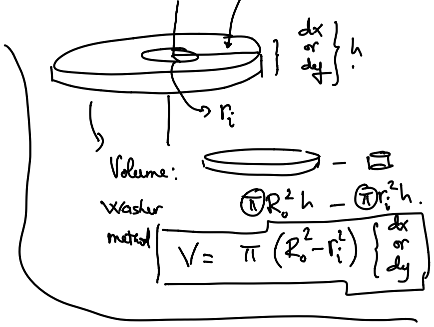
from $y = x^3 \Rightarrow x = \sqrt[3]{y}$

$$V = \pi \int_0^1 \left(\sqrt[3]{y}\right)^2 \cdot dy = \pi \int_0^1 y^{2/3} dy$$

$$= \pi \cdot \frac{3}{5} \cdot y^{5/3} \Big|_0^1 = \boxed{\frac{3\pi}{5}}$$

c) $y=\sqrt{x}$ and $y=x$ about $x=2$





$$V = \pi\left(R_o^2 - r_i^2\right)\left\{\begin{matrix} dx \\ \text{or} \\ dy \end{matrix}\right.$$

x=2

$\}dy$. $V = \pi\left(R_o^2 - r_i^2\right)\left\{\begin{matrix} dx \\ \text{or} \\ dy \end{matrix}\right.$

where $R_o = \;$ ($x=2$) $= x_r - x_L = 2 - y^2$

$r_i = \cancel{1-x}$ ($x=2$) $= x_r - x_L = 2 - y$

$x=y$

$$V = \pi\int_0^1 \left((2-y^2)^2 - (2-y)^2\right)dy.$$

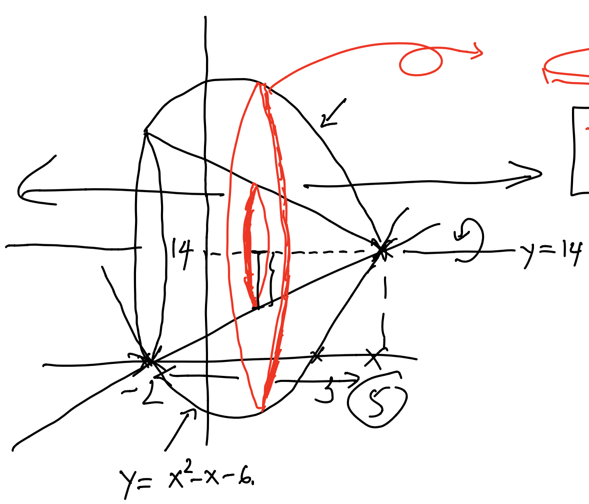
sketch & set up the integrals.

d) $y = x^2 - x - 6$ and $y = 2x+4$ about $y=14$

$= (x+2)(x-3)$ $\Rightarrow$ pts of intersection. $\Rightarrow$ $x^2 - x - 6 = 2x+4$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$ $\Rightarrow$ $x = 5, -2$.



$\}dx$

$$\boxed{V = \pi\left[R_o^2 - r_i^2\right]dx}.$$

where $R_o = \;$ ($y=14$, $y=x^2-x-6$) $= y_{top} - y_{bot} = 14 - (x^2 - x - 6)$

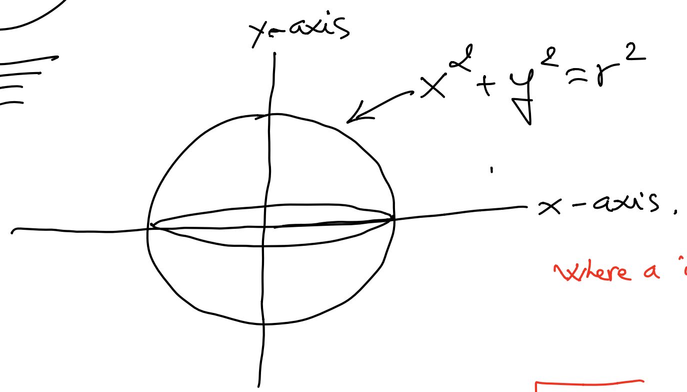
$= 20 - x^2 + x$

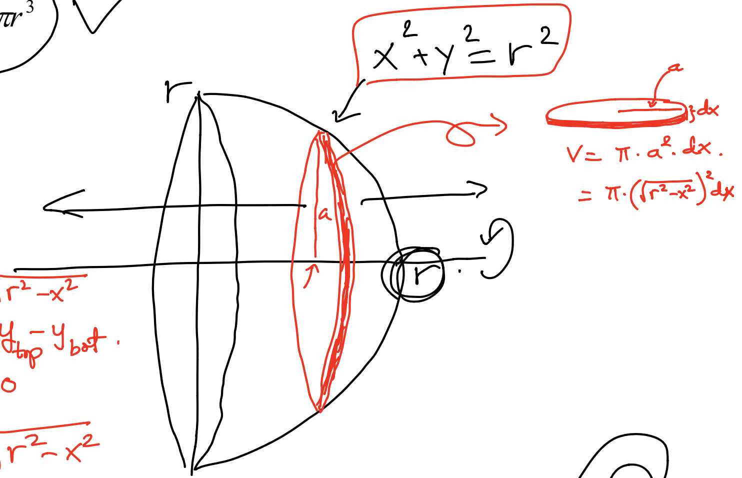
$r_i = \;$ ($y=14$, $y=2x+4$) $= y_{top} - y_{bot} = 14 - (2x+4)$

$= 10 - 2x$.

$$V = \int_{-2}^{5} \pi\left[\left(20 - x^2 + x\right)^2 - \left(10 - 2x\right)^2\right]dx$$

Ex: Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$





where a is $y_{top} - y_{bot}$.

$y_{top} = \sqrt{r^2-x^2}$

$y_{bot} = 0$

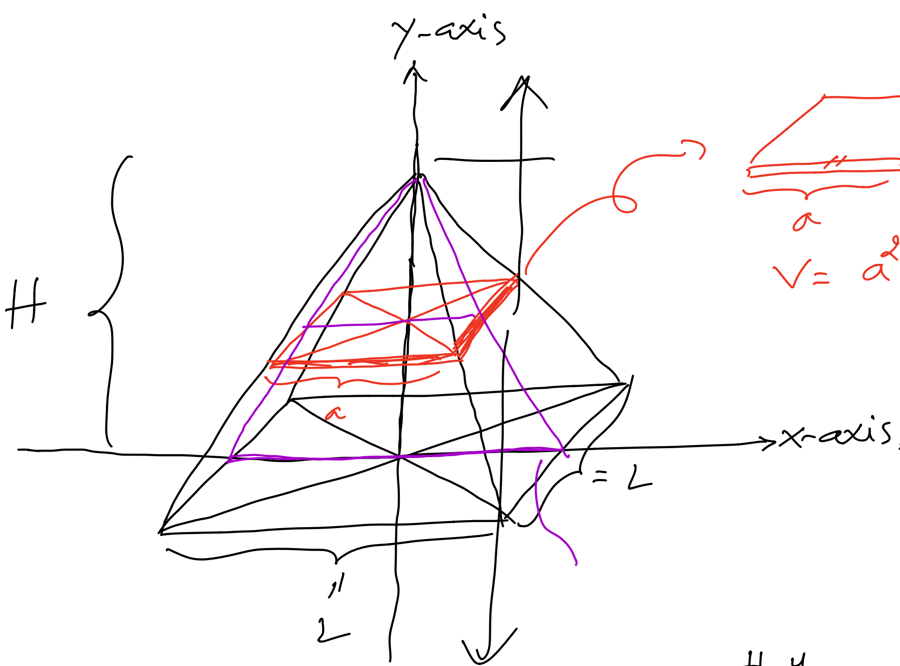
$a = \sqrt{r^2-x^2} - 0 = \sqrt{r^2-x^2}$

$$V = 2\int_0^r \pi \left(\sqrt{r^2-x^2}\right)^2 dx = 2\pi \int_0^r (r^2-x^2)\,dx = 2\pi \left[r^2x - \tfrac{1}{3}x^3\right]_0^r$$

$$= 2\pi\left[r^3 - \tfrac{1}{3}r^3\right] = 2\pi \cdot \tfrac{2}{3}r^3 = \boxed{\tfrac{4}{3}\pi r^3}$$

Parallel Cross Sections

Ex: Find the volume of a pyramid whose base is a square with side L and whose height is H.



$V = a^2 dy$

$$= \int_0^H \left(\frac{L}{H}(H-y)\right)^2 dy.$$

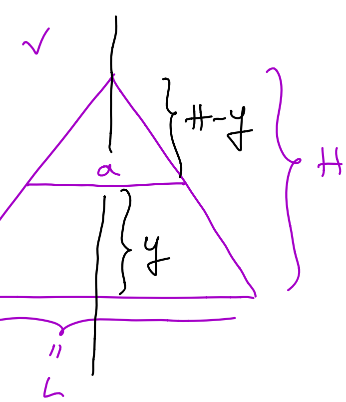
$$= \frac{L^2}{H^2}\int (H^2 - 2Hy + y^2)\,dy.$$

$$= \frac{L^2}{H^2}\left[H^2y - Hy^2 + \tfrac{1}{3}y^3\right]_0^H$$

$$= \frac{L^2}{\cancel{H^2}}\left[\cancel{H^3} - \cancel{H^3} + \tfrac{1}{3}H^3\right] = \boxed{\tfrac{1}{3}L^2H}$$

Similar Δ: $\dfrac{a}{L} = \dfrac{H-y}{H}$.

$a = \frac{L}{H}(H-y)$

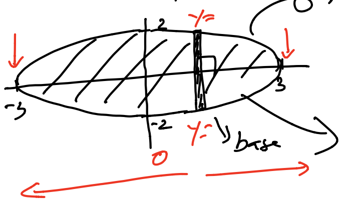


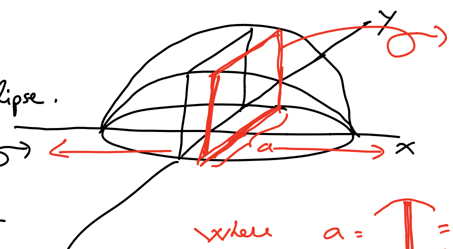
base

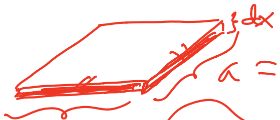
Ex: The base of a solid is bounded by $\{(x,y) \mid 4x^2+9y^2 \le 36\}$. All parallel cross sections are squares which perpendicular to the base and the x – axis. Find the volume of the solid.

$$\frac{4x^2}{36}+\frac{9y^2}{36}\le\frac{36}{36}$$

$$\boxed{\frac{x^2}{9}+\frac{y^2}{4}\le 1} \leftarrow \text{Ellipse.}$$







$$a = 4\sqrt{1-\frac{x^2}{9}}$$

$$V = a^2\,dx.$$

where $a = y_{top} - y_{bot.} = \sqrt{4(1-\frac{x^2}{9})} + \sqrt{4(1-\frac{x^2}{9})}$
$= 2\left(\sqrt{4(1-\frac{x^2}{9})}\right) = 4\sqrt{1-\frac{x^2}{9}}$

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

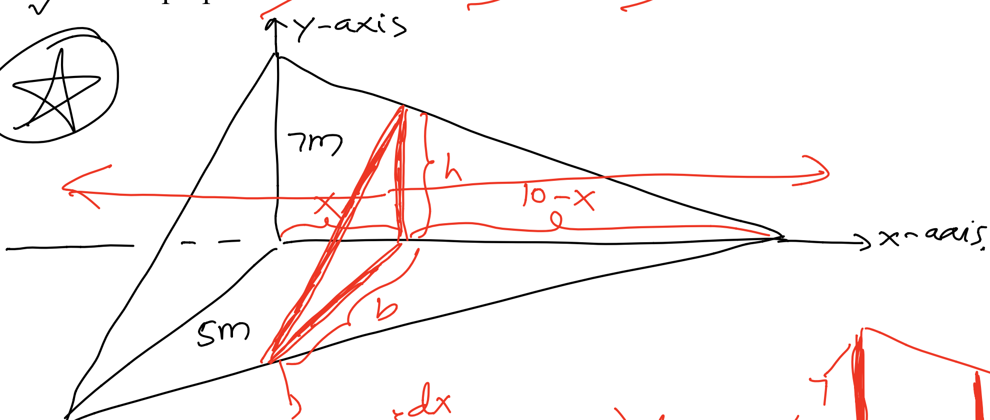
$$\frac{y^2}{4} = 1-\frac{x^2}{9} \Rightarrow y^2 = 4\left(1-\frac{x^2}{9}\right) \Rightarrow y = \pm\sqrt{4\left(1-\frac{x^2}{9}\right)}.$$

$$V = \int_{-3}^{3}\left(4\sqrt{1-\frac{x^2}{9}}\right)^2 dx = 16\int_{-3}^{3}\left(1-\frac{x^2}{9}\right)dx = 16\left[x-\frac{1}{27}x^3\right]_{-3}^{3}.$$

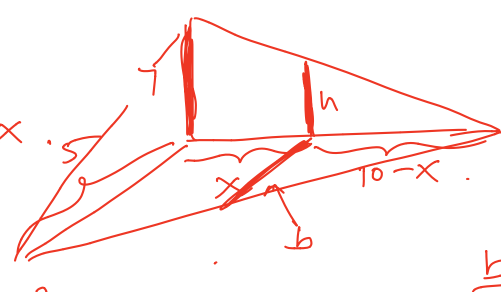
$$= 16\left[(3-1)-(-3+1)\right]$$

$$= 16\left[2+2\right] = 16(4) = \boxed{64}$$

Ex: The base of a solid is bounded by $y = x^2$ and $y = x$. All parallel cross sections are semi-circle perpendicular to the base and the x-axis.



$$V = \left(\frac{1}{2}bh\right)dx.$$



$$\frac{h}{7} = \frac{10-x}{10}$$

$$h = \frac{7}{10}(10-x)$$

$$\frac{b}{5} = \frac{10-x}{10}$$

$$b = \frac{1}{2}(10-x)$$

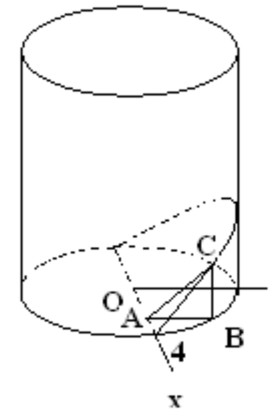
$$V = \int_0^{10}\boxed{\frac{1}{2}\cdot\frac{1}{2}(10-x)\cdot\frac{7}{10}(10-x)\cdot dx}.$$

$$= \frac{7}{40}\int_0^{10}\left(100-20x+x^2\right)dx = \frac{7}{40}\left[100(10) - 10(10)^2 + \frac{1}{3}(10)^3\right]$$

$$= \boxed{\frac{7}{40}\cdot\frac{1000}{3}} = \frac{7000}{120} = \frac{350}{6}$$

Total Volume $\longrightarrow$ $\boxed{\frac{175}{3} m^3}$

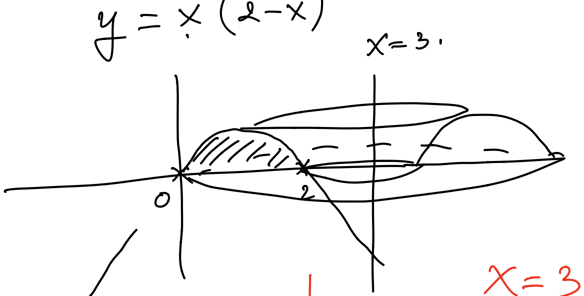
Ex: A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30 degree along the diameter of the cylinder. Find the volume of the wedge.

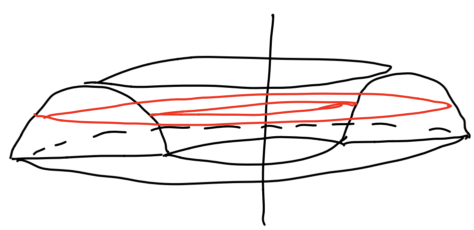


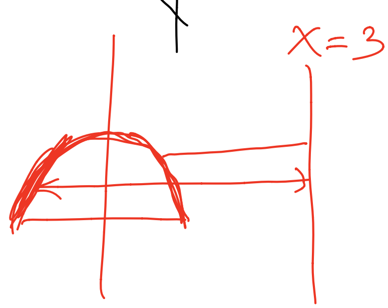
6.3 ***Volume by cylindrical shell method.***

$\underbrace{y = 2x - x^2 \ \& \ y = 0}$; about $\underline{x = 3.}$

$y = x(2-x)$







$\pi \cdot (R_o^2 - r_o^2) \left\{ \frac{dx}{dy} \right.$

$\parallel$

0

Washer method fails,

When R_o & r_i are controled by the same curve

Ex The region bounded by the following curves, sketch and set up integrals for volume:

a) $y = 6 + x - x^2$ and $y = 2x - 6$

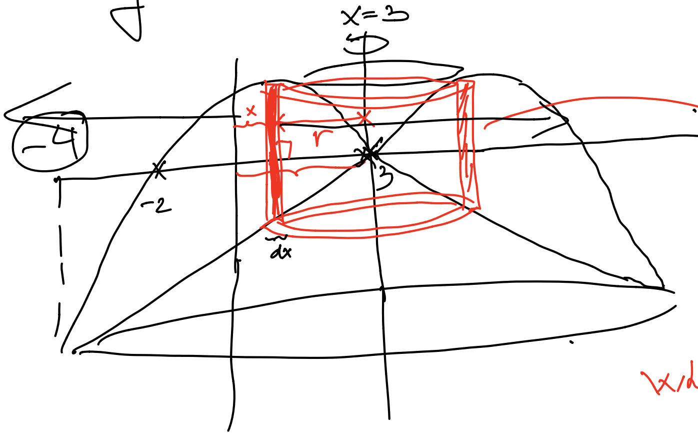
i) rotates about the line $x = 3$

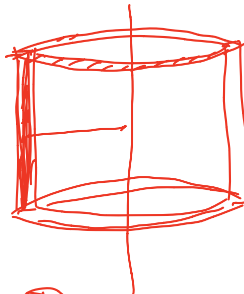
pts of intersection $\Rightarrow 6 + x - x^2 = 2x - 6$

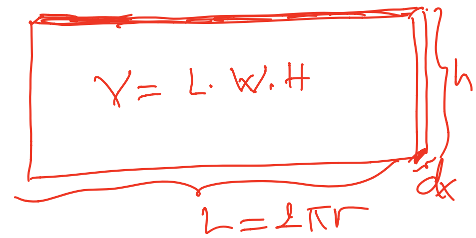
$x^2 + x - 12 = (x+4)(x-3) = 0$

$x = -4, 3$.

$y = 6 + x - x^2 = (2 + x)(3 - x)$







where $H: = y_{top} - y_{bot} = 6 + x - x^2 - (2x - 6) = -x^2 - x + 12$.

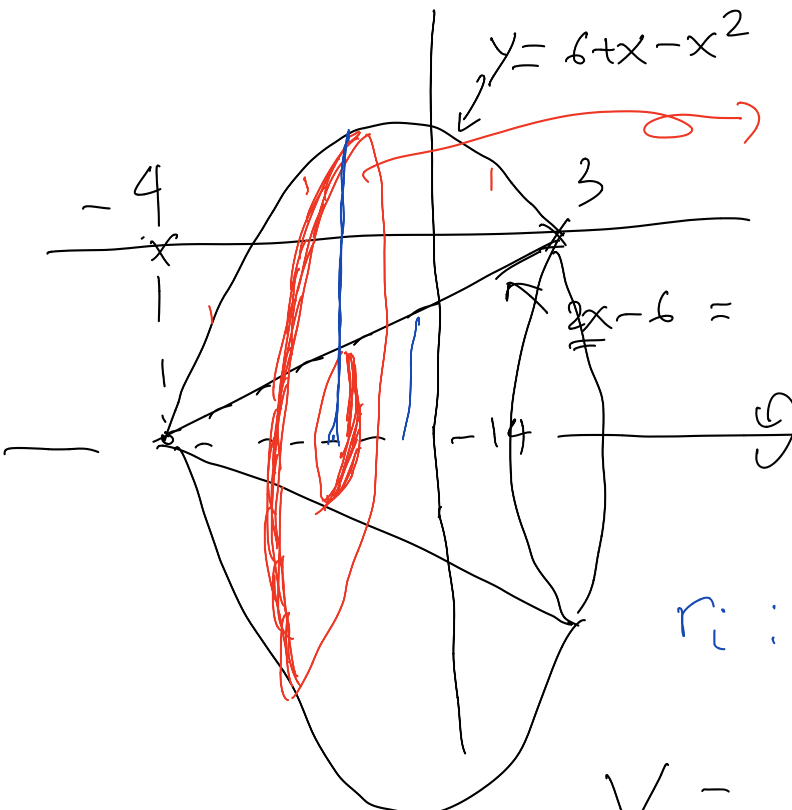
$L = 2\pi(3 - x)$

$$V = 2\pi \int_{-4}^{3} (3 - x)(-x^2 - x + 12)\,dx$$

Note: washer $\Rightarrow$ disc $\perp$ to the axis of rotation.

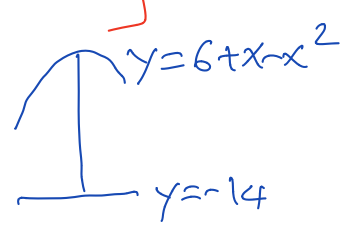
shell $\Rightarrow$ rectangle // to the axis of rotation

ii) rotates about the line $y = -14$



$V = \pi\left[R_o^2 - r_i^2\right]dx$

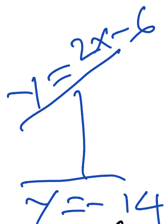
where R_o:



$\Rightarrow R_o = 6 + x - x^2 - (-14)$

$= 20 + x - x^2$

r_i:



$= 2x - 6 + 14 = 2x + 8$

$$V = \pi \int_{-4}^{3} \left[(20 + x - x^2)^2 - (2x+8)^2\right] dx$$

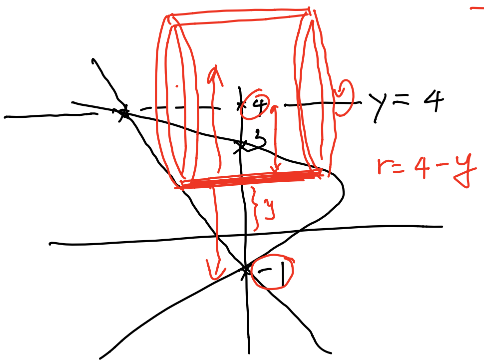
(b) The region bounded by $x=-y^2+2y+3$ and $x+y=-1 \Rightarrow x=-1-y$

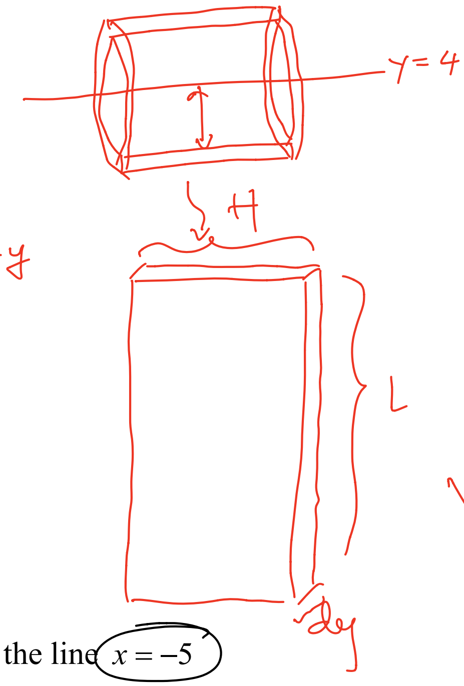
i) rotated about the line $y=4$

$$x=-(y^2-2y-3)=-(y-3)(y+1)$$

pts of intersection: $-y^2+2y+3=-1-y$

$$y^2-3y-4=0$$
$$(y-4)(y+1)=0$$
$$y=-1,4.$$

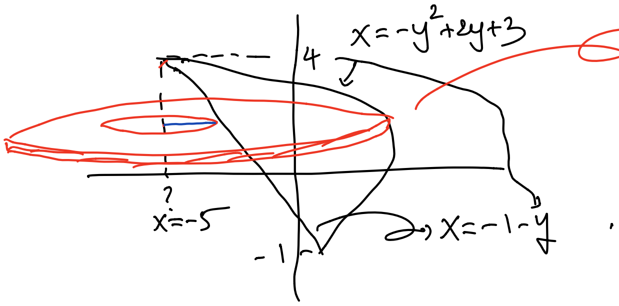




$H:$ $=x_R-x_L$

$$H=-y^2+2y+3-(-1-y)$$
$$=-y^2+3y+4.$$
$$L=2\pi r = 2\pi(4-y)$$
$$V=2\pi\int_{-1}^{4}(4-y)(4+3y-y^2)\,dy =$$

ii) rotated about the line $x=-5$



$$V=\pi\left[R_o^2-r_i^2\right]dy.$$

Where: $R_o=$ (from $x=-5$ to $x=-y^2+2y+3$)

$$=-y^2+2y+3+5=8+2y-y^2.$$

$r_i=$ (from $x=-5$ to $x=-1-y$)

$$=-1-y+5=4-y.$$

$$V=\int_{-1}^{4}\pi\left[(8+2y-y^2)^2-(4-y)^2\right]dy$$