unit for work: $W = F\cdot d$.

F: Newton: N ; d = meters: m ⇒ W = Nm = Joule.

F = Pounds: lb ; d = feet: ft ⇒ W = ft-lb.

Section 6.4 **Work**

Work Done by a Constant Force: When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we calculate the work W done by the force on the body with the formula: **Work = Force x Distance.** **W = Fd**

Different kind of forces such as push, pull, lift…but it's also about weight (Weight = mass x gravity ➔ F=mg)
Note: Gravity = 9.8 m/second square = 32 ft/second square.

Note: If you jack up the side of a 2000 lb car 1.25 ft to change a tire (you have to apply a constant vertical force about 1000 lb), you will perform 1000(1.25) = 1250 ft-lb of work on the car. In SI units, you have applied a force of 4448 N through a distance of 0.381 m to do 4448(0.381)=1695 J of work.

Ex: How much work is done in lifting a 3 kg (mass) book off the floor to put it on a desk that is 0.8 m high?
$F = 3(9.8)$ ➔ $W = Fd = 3(9.8)(.8) = 23.52$ kg(m)/sec^2 = 23.52 ~~N~~ J.

Ex: How much work is done in lifting a 3 lb book off the floor to put it on the desk that is 0.8 ft high?
W =Fd = 3 (.8) = 3.2 ft – lb. (Mention that as long as the unit in pound it's already taken the gravity into consideration)

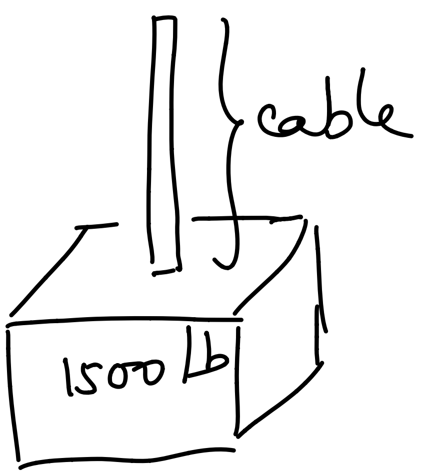
So far we have dealt with a constant force such as gravity, what happens if the force change is changing. i.e. force is a function depend on other factors, for example, pulling a robe to the top of a building. Imagine a robe is hanging over the edge from the top of a building, how much work we need to pull up the robe. Obviously, the more we pull up the robe, the less weight the robe is, i.e. the force that gravity exert on the robe is depending on how much we already pulled up.

Let force be a function $f(x)$. Then the work $W \approx f(x)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

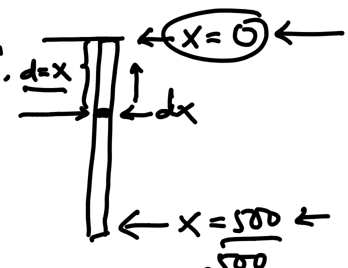
Work: 1. cable + elevator 2. spring. 3. tank of water.

Ex: How much work is required to pull a 500 – ft – cable that weight 250 lb and it attach to a 1,500 – lb – elevator.



For the elevator: $W_e = F_e \cdot d_e = (1500\,lb)(500\,ft) = 750{,}000$ ft-lb.

For the cable:



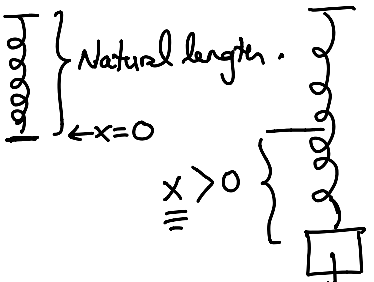
L	F
500 ft	250 lb.
÷500	÷500
1 ft	$\frac{1}{2}$ lb.
$\frac{1}{2}$ ft	$\frac{1}{2}\cdot(\frac{1}{2})$ lb.
$\frac{1}{1000}$ ft	$\frac{1}{2}(\frac{1}{1000})$ lb
dx →	$\frac{1}{2}$ dx

$$\begin{cases} d = x \\ F = \frac{1}{2}dx \end{cases} \Rightarrow W_c = F\cdot d = \int_0^{500} x\cdot\frac{1}{2}dx = \frac{1}{4}x^2\Big|_0^{500} = \frac{1}{4}(500)^2$$

Total work = $W = W_e + W_c$
$= 750{,}000$ (ft – lb) + 62,500 (ft – lb)

$$= 812{,}500 \text{ ft-lb}$$

Spring:



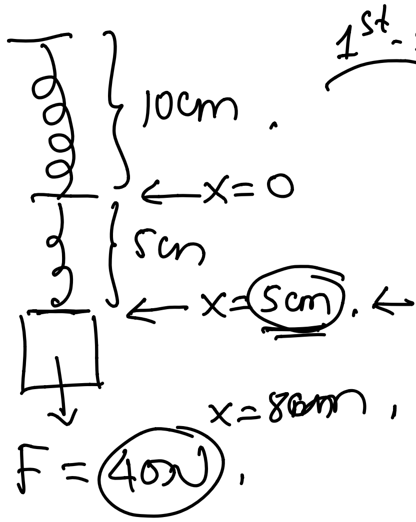
Hook's Law: $\boxed{\begin{array}{c} F = kx \\ W = \int kx\,dx \end{array}}$

1'st: find the constant k

(mass.)(gravity) = force.

$$F = mg$$

Ex: A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?



1st-step: Find the spring constant k:

$$F = kx$$

$$40N = k(5cm) = k(0.05m)$$

$$k = \frac{40}{0.05} = 800.$$

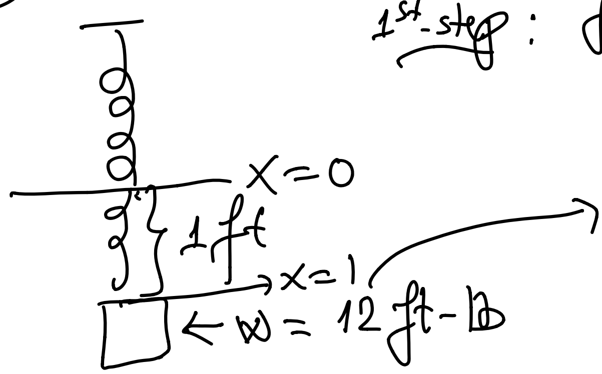
$$W = \int kx\,dx = \int_{5cm=0.05m}^{8cm=0.08m} 800x\,dx = 400x^2 \Big|_{0.05}^{0.08}$$

$$= 400\left[(0.08)^2 - (0.05)^2\right]$$

$$= 1.56 J.$$

Ex: If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 in. beyond its natural length?

1st-step: find the spring constant k:

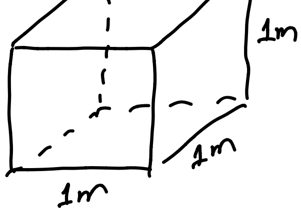


$$W = \int kx\,dx.$$

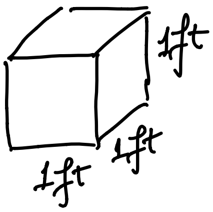
$$12 = \int_0^1 kx\,dx = \frac{1}{2}kx^2 \Big|_0^1 = \frac{1}{2}k$$

$$\Rightarrow k = 24$$

$$W = \int kx\,dx = \int_0^{9'' = \frac{9}{12}' = \frac{3}{4}ft} 24x\,dx = 12x^2 \Big|_0^{\frac{3}{4}} = 12\left(\frac{3}{4}\right)^2 = 12 \cdot \frac{9}{16} = \frac{27}{4} \text{ ft-lb.}$$

Water: 

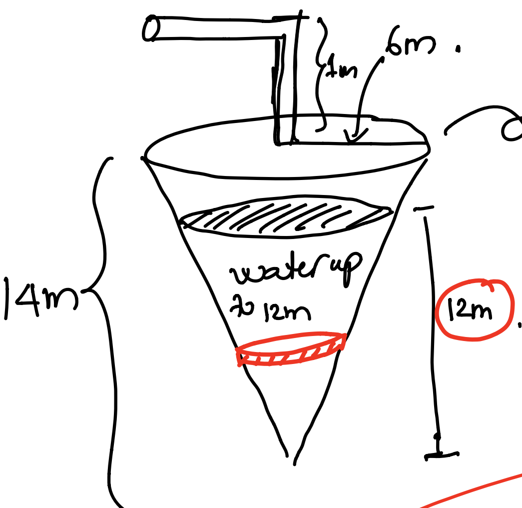
Force: $F = (1000kg)(9.8 m/sec^2) = 9800 N/m^3$.



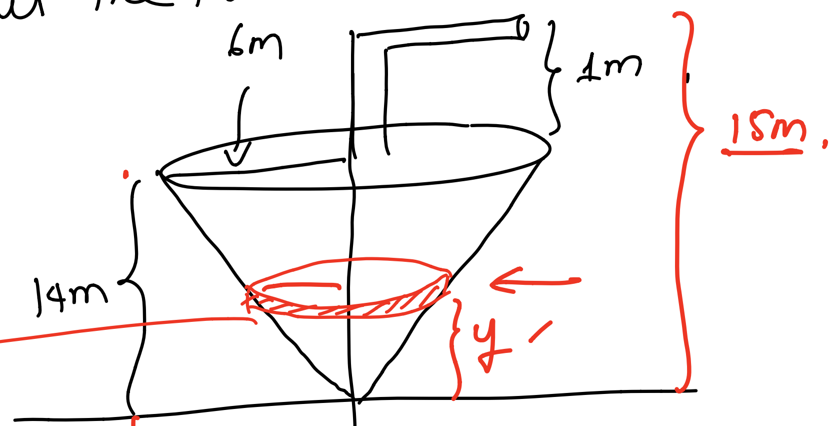
Force: $F = 62.5\ lb/ft^3$.

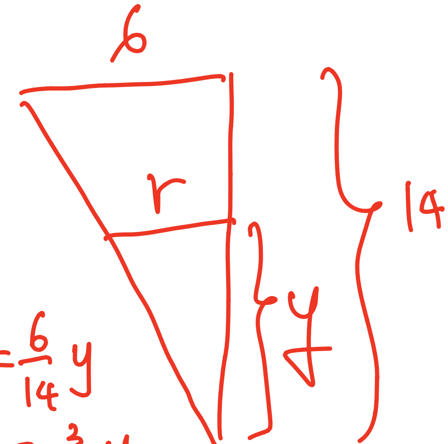
Given a volume $V \Rightarrow$ Force $F = \begin{cases} 9800V \\ \text{or} \\ 62.5V \end{cases}$

Ex: Find the work to pump out the water:



Put the tank onto the xy-coord.





$V = \pi \cdot r^2 \cdot dy$ where $\frac{r}{6} = \frac{y}{14} \Rightarrow r = \frac{6}{14}y = \frac{3}{7}y$.

1. $V = \pi\left(\frac{3}{7}y\right)^2 dy$
2. $F = 9800 \cdot \pi \cdot \frac{9}{49} \cdot y^2 \cdot dy$.
3. $d = 15 - y$
4. $W = F \cdot d = \frac{9800\pi \cdot 9}{49}\int_0^{12} y^2(15-y)\,dy = \frac{9800 \cdot 9\pi}{49}\left[5(12)^3 - \frac{1}{4}(12)^4\right]$

$W = 9{,}771{,}609.\frac{79}{xx}\ J.$

Ex: Suppose that it took 20 years to construct the great pyramid of Khufu at Gizeh, Egypt. This pyramid is 500ft high and has a square base with edge length 750ft. Suppose also that the pyramid is made of rock with density $\rho = 120 lb/ft^3$. Finally, suppose that each laborer did 160 ft lb/h of work in lifting rock from ground level to their final position in the pyramid and worked 12 hours daily for 330 days/yr. How many laborers would have been required to construct the pyramid?

Sol: $\dfrac{s}{750} = \dfrac{500-y}{500} \Rightarrow s = \dfrac{3}{2}(500-y)$

The cross sectional area at the height y is $A(y) = \dfrac{9}{4}(500-y)^2$

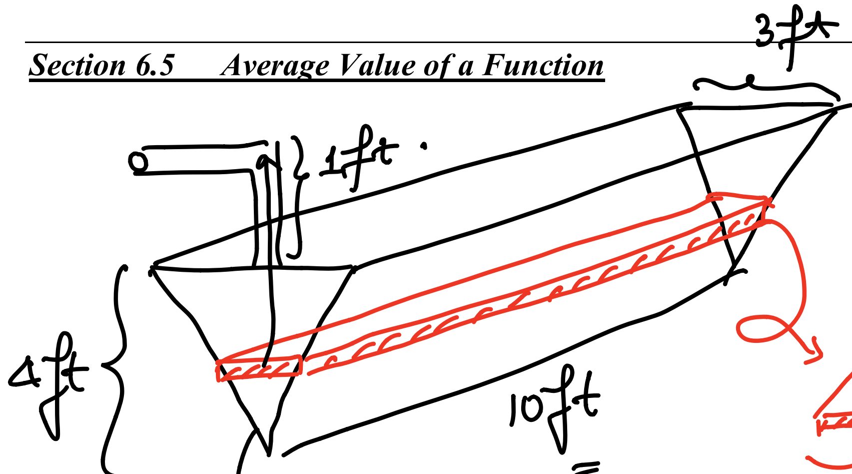
The volume of the one single slice is $V(y) = \dfrac{9}{4}(500-y)^2\,dy$

Therefore the work for one single slice would be $120(y)\dfrac{9}{4}(500-y)^2 = 270\left(250{,}000y - 1000y^2 + y^3\right)dy$

Total work is $W = \int_0^{500} 270\left(250{,}000y - 1000y^2 + y^3\right)dy \approx 1.406\times 10^{12}\ ft\cdot lb$

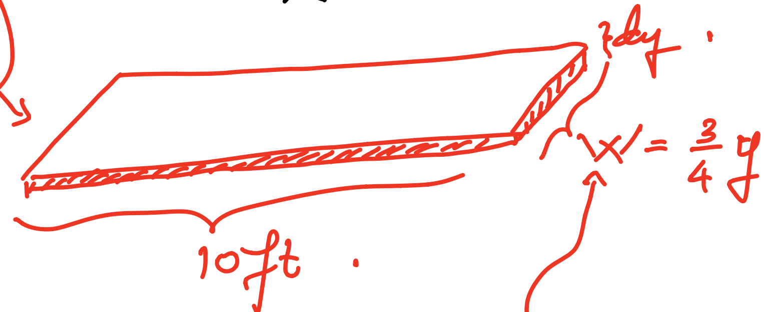
Because each laborer does $(160)(12)(330)(20) \approx 1.267\times 10^{7}\ ft\cdot lb$ of work, the construction of the pyramid would under our assumptions, have required $\dfrac{1.406\times 10^{12}}{1.267\times 10^{7}} \approx 111{,}000$ laborers.

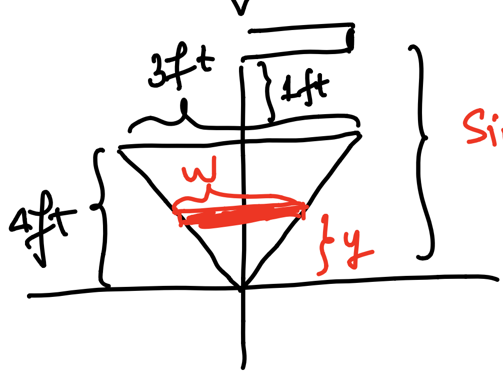
Section 6.5 Average Value of a Function



full of water.

Work = ?





Similar Δ: $\dfrac{w}{3} = \dfrac{y}{4} \Rightarrow w = \dfrac{3}{4}y$

$V = 10\left(\frac{3}{4}y\right)dy = \frac{15}{2}y\,dy$.

$F = 62.5V = (62.5)\left(\frac{15}{2}\right)y\,dy = 468.75y\,dy$

$d = 5-y$.

$W = F\cdot d = \int_0^4 468.75y(5-y)\,dy$.

$= 468.75\left[\frac{5}{2}(4)^2 - \frac{1}{3}(4)^3\right] = \underline{8750\ ft-lb}$.

Ex: Find the average value of the following function over a given interval.

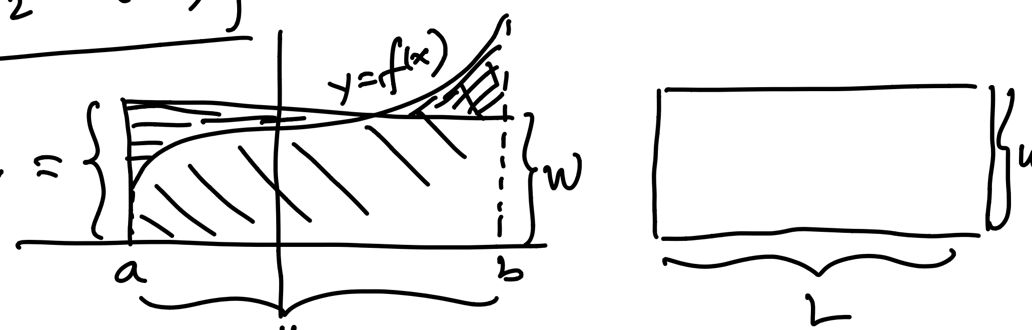
a) $f(x)=\dfrac{3}{2x+1}$; $[\overset{a}{1},\overset{b}{2}]$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-1}\int_1^2 \frac{3}{2x+1}\,dx = \int_1^2 \frac{3}{2x+1}\,dx$$

$$= \frac{3}{2}\ln|2x+1|\Big|_1^2 = \frac{3}{2}\left[\ln 5 - \ln 3\right]$$

$$\boxed{f_{avg} = \frac{3}{2}\ln\left(5/3\right)}$$

f_{avg}



$L = (b-a)$

Area of ▭ $= L\,w = \int_a^b f(x)\,dx$

$$(b-a)w = \int_a^b f(x)\,dx$$

Average value of $f(x)$: $f_{avg} = w = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

b) $f(x)=\dfrac{5x^4}{x^5+7}$; $[0,2]$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-0}\int_0^2 \frac{5x^4}{x^5+7}\,dx$$

let $u = x^5+7 \Rightarrow du = 5x^4\,dx$

$x=0 \Rightarrow u=7$

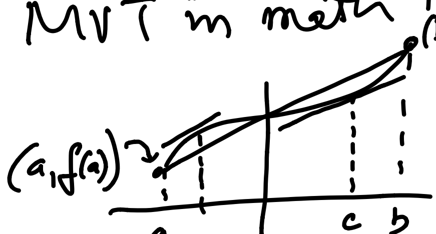
$x=2 \Rightarrow u = 2^5+7 = 39$.

$$= \frac{1}{2}\int_7^{39}\frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_7^{39}$$

$$= \frac{1}{2}\left[\ln(39) - \ln(7)\right]$$

$$= \boxed{\frac{1}{2}\ln\left(\frac{39}{7}\right)}$$

MVT in math 180



$m = \dfrac{f(b)-f(a)}{b-a}$

$\exists\, c \in (a,b)$ s.t.

$f'(c) = m = \dfrac{f(b)-f(a)}{b-a}$.

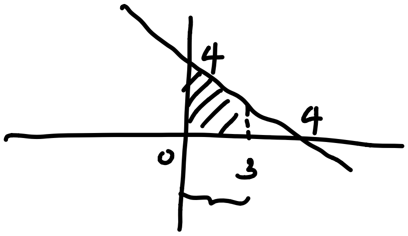
$\Rightarrow \underline{(b-a)\cdot f'(c) = f(b)-f(a)}$

$c \in (a, b)$

The Mean Value Theorem for Integrals: If f is continuous on [a, b], then there exists a number c [a, b] such that

$$\boxed{\int_a^b f(x)dx = f(c)(b-a)} \Rightarrow f(c) = \frac{1}{b-a}\int_a^b f(x)dx = f_{avg}.$$

Ex: Find the average value of $f(x) = 4 - x$ on [0, 3] and where f actually takes on this value at some point in the given domain.



$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{3-0}\int_0^3 (4-x)dx$$

$$= \frac{1}{3}\left[4x - \frac{1}{2}x^2\right]_0^3 = \frac{1}{3}\left[12 - \frac{9}{2}\right].$$

$$f_{avg} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{15}}{2} = \frac{5}{2}$$

$\exists\, c \in (0,3)$ such that $f(c) = f_{avg} = \frac{5}{2}$

$$\Rightarrow f(c) = 4 - c = \frac{5}{2} \Rightarrow c = 4 - \frac{5}{2} = \frac{3}{2}.$$