Ex: Evaluate the integral: $\int_0^2 \frac{1}{(x-1)^2}\,dx =$

Let $u = x - 1$ $\begin{cases} x = 2 \Rightarrow u = 1 \\ x = 0 \Rightarrow u = -1 \end{cases}$

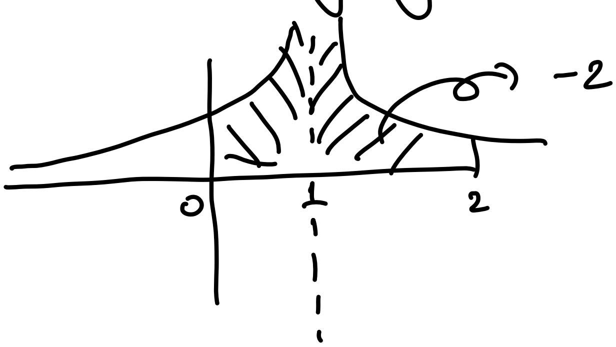
$du = dx$

$$\int_{-1}^{1} \frac{1}{u^2}\,du = \int_{-1}^{1} u^{-2}\,du = \frac{u^{-1}}{-1}\Bigg|_{-1}^{1} \leftarrow$$

$$= -\left[1 - (-1)\right] = -2.$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

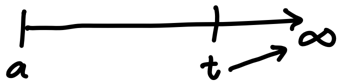
Let's check the graph: $y = f(x) = \frac{1}{(x-1)^2}$



Type 1: Improper Integrals (Infinite Limits of Integration)

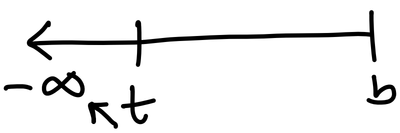
1. If f is continuous on the interval $[a, \underline{\underline{\infty}}]$

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$



2. If f is continuous on the interval $(-\infty, b]$

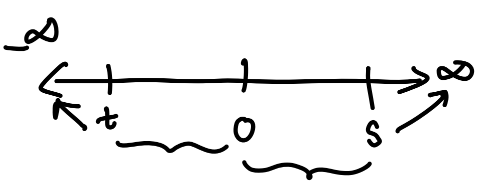
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$



3. If f is continuous on the interval $(-\infty, \infty)$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx.$$

$$= \lim_{t\to-\infty}\int_t^0 f(x)\,dx + \lim_{s\to\infty}\int_0^s f(x)\,dx$$



Ex: Evaluating the following improper integrals:

a) $\int_0^\infty \frac{x}{\left(4x^2+1\right)^{5/2}}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(4x^2+1)^{5/2}}\,dx$

$$\begin{cases} \text{let } u = 4x^2+1 \\ du = 8x\,dx \\ \frac{du}{8} = x\,dx \end{cases}$$

$0 \quad t \to \infty$

$$= \lim_{t\to\infty}\int \frac{\frac{du}{8}}{u^{5/2}} = \frac{1}{8}\lim_{t\to\infty}\int u^{-5/2}\,du.$$

$$= \frac{1}{8}\lim_{t\to\infty} u^{-3/2}\left(-\frac{2}{3}\right) = -\frac{1}{12}\lim_{t\to\infty}\left(4x^2+1\right)^{-3/2}\Big|_0^t$$

$$= -\frac{1}{12}\lim_{t\to\infty}\left[\frac{1}{\cancel{(4t^2+1)^{3/2}}_{\to\infty}} - 1\right] = -\frac{1}{12}(-1) = \frac{1}{12}$$

"Convergent"

b) $\int_1^\infty \frac{x^2}{\sqrt[3]{7x^3+1}}\,dx = \lim_{t\to\infty}\int_1^t \frac{x^2}{\sqrt[3]{7x^3+1}}\,dx$

$$\begin{cases} \text{let } u = \sqrt[3]{7x^3+1} \\ u^3 = 7x^3+1 \\ 3u^2\,du = 21x^2\,dx \Rightarrow \frac{1}{7}u^2\,du = x^2\,dx \end{cases}$$

$1 \quad t \quad \infty$

$$= \lim_{t\to\infty}\int \frac{\frac{1}{7}u^{\cancel{2}}\,du}{\cancel{u}} = \frac{1}{7}\lim_{t\to\infty}\int u\,du. = \frac{1}{7}\lim_{t\to\infty}\frac{u^2}{2}.$$

$$= \frac{1}{14}\lim_{t\to\infty}\left(\sqrt[3]{7x^3+1}\right)^2\Big|_1^t$$

$$= \frac{1}{14}\lim_{t\to\infty}\left[\left(7t^3+1\right)^{2/3} - 4\right] = \infty \quad \text{"Divergent"}$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$

$$\longrightarrow \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

c) $\int_{-\infty}^{\infty} \frac{x}{x^4+9}dx = \lim_{t\to-\infty} \int_t^0 \frac{x}{x^4+9}dx + \lim_{s\to\infty} \int_0^s \frac{x}{x^4+9}dx$

$$\begin{cases} u = x^2 \Rightarrow x^4 = u^2 \\ du = 2x\,dx \\ \frac{du}{2} = x\,dx. \end{cases}$$

$-\infty \quad t \quad 0 \quad s \quad \infty$

$$= \lim_{t\to-\infty} \int \frac{\frac{du}{2}}{u^2+9} + \lim_{s\to\infty} \int \frac{\frac{du}{2}}{u^2+9.} = \frac{1}{2}\left[\lim_{t\to-\infty} \int \frac{du}{u^2+9} + \lim_{s\to\infty} \int \frac{du}{u^2+9}\right]$$

$$= \frac{1}{2}\left[\lim_{t\to-\infty} \cdot \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + \lim_{s\to\infty} \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]$$

$$= \frac{1}{6}\left[\lim_{t\to-\infty} \tan^{-1}\left(\frac{x^2}{3}\right)\Big|_t^0 + \lim_{s\to\infty} \tan^{-1}\left(\frac{x^2}{3}\right)\Big|_0^s\right]$$

$$= \frac{1}{6}\left[\lim_{t\to-\infty}\left(0 - \tan^{-1}\left(\frac{t^2}{3}\right)\right) + \lim_{s\to\infty}\left(\tan^{-1}\left(\frac{s^2}{3}\right) - 0\right)\right] = \frac{1}{6}\left[\left(0 - \cancel{\frac{\pi}{2}}\right) + \left(\cancel{\frac{\pi}{2}} - 0\right)\right]$$

$= 0$ "Convergent"

d) $\int_{-\infty}^{0} \frac{x}{e^{1+3x^2}} dx = \lim_{t\to-\infty} \int_t^0 \frac{x}{e^{1+3x^2}}dx$

$$\begin{cases} u = 1+3x^2 \\ du = 6x\,dx \\ \frac{du}{6} = x\,dx \end{cases}$$

$\int e^{kx}dx = \frac{e^{kx}}{k}$

$-\infty \quad t \quad 0$

$$= \lim_{t\to-\infty} \int \frac{\frac{du}{6}}{e^u} = \frac{1}{6}\lim_{t\to-\infty} \int e^{-u} du = \frac{1}{6}\lim_{t\to-\infty} \frac{e^{-u}}{-1}$$

$$= -\frac{1}{6}\lim_{t\to-\infty} e^{-(1+3x^2)}\Big|_t^0 = -\frac{1}{6}\lim_{t\to-\infty}\left[e^{-1} - e^{-(1+3t^2)}\right]$$

$$= -\frac{1}{6}\lim_{t\to-\infty}\left[\frac{1}{e} - \cancel{\frac{1}{e^{1+3t^2}}}\right] = -\frac{1}{6e} \quad \text{"Convergent"}$$

$t^2 \to \infty$

P-Test Theorem: For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

$$\underline{\text{Case 1}}\text{ if } p=1 \Rightarrow \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t$$

$$= \lim_{t\to\infty}\Big[\ln|t| - \underbrace{\ln 1}_{0}\Big] = \infty : \text{divergent}$$

$$\underline{\text{Case 2}}: \text{ if } p\neq 1 . \int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t$$

$$= \frac{1}{1-p}\lim_{t\to\infty}\left(t^{-p+1} - 1\right) = \frac{1}{1-p}\left[\infty^{-p+1} - 1\right]$$

For convergence $\Rightarrow -p+1<0 \Rightarrow \boxed{p>1}$

$$\boxed{\underline{p\text{-Test}}: \int_{\underline{a>0}}^\infty \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p\leq 1 \end{cases}}$$

Ex: Test for <u>convergence</u> / <u>divergence</u>:

a) $\int_1^\infty \frac{\sqrt{x^5}}{x^3}\,dx = \int_1^\infty \frac{1}{x^p}\,dx$

$$\int_1^\infty \frac{x^{5/2}}{x^3}\,dx = \int_1^\infty \frac{1}{x^{3-5/2}}\,dx$$

$$= \int_1^\infty \frac{1}{x^{1/2}}\,dx$$

$p = \frac{1}{2} < 1 \Rightarrow$ Divergent

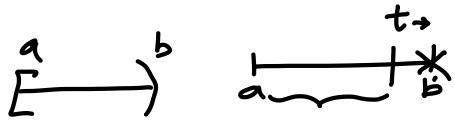
b) $\int_1^\infty \frac{x}{\sqrt[3]{x^7}}\,dx = \int_1^\infty \frac{1}{x^p}\,dx$

$$\int_1^\infty \frac{1}{x^{7/3-1}}\,dx$$

$$= \int_1^\infty \frac{1}{x^{4/3}}\,dx$$

$p = \frac{4}{3} > 1 \Rightarrow$ Convergent.

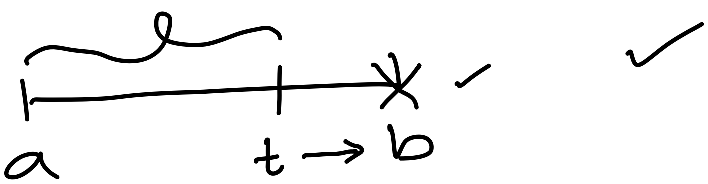
Type 2: ***Discontinuous Integrands***



1. If f is continuous on the interval [a, b) and approaches infinity at b.

$f(x)$ is discontinuous at $x = b$. (or. $f(b)$ = undefined).

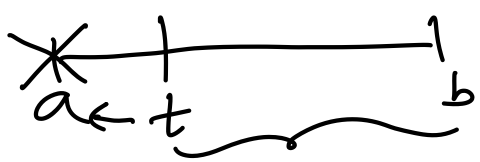
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$



2. If f is continuous on the interval (a, b] and approaches infinity at a.

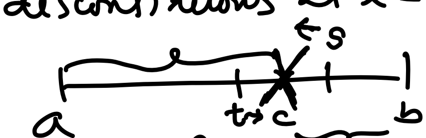
If $f(x)$ is discontinuous at $x = a$ (or $f(a)$ = undefined)

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$



3. If f is continuous on the interval [a,b], except for some c in (a, b).

$f(x)$ is discontinuous at $x = c$ (or $f(c)$ = undefined)



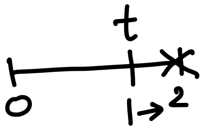
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{s \to c^+} \int_s^b f(x)\,dx$$

Ex: Evaluate the following improper integrals

a) $\int_0^2 \frac{x^2}{\sqrt[3]{x^3-8}}\,dx = f(x)$ $= \lim_{t\to 2^-} \int_0^t \frac{x^2}{\sqrt[3]{x^3-8}}\,dx$

$$\begin{cases} u = \sqrt[3]{x^3-8} \\ u^3 = x^3-8 \\ 3u^2\,du = 3x^2\,dx \\ u^2\,du = x^2\,dx \end{cases}$$



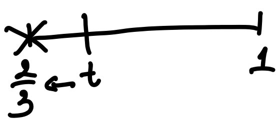
$$= \lim_{t\to 2^-} \int \frac{u^{\cancel{2}}\,du}{\cancel{u}} = \lim_{t\to 2^-} \frac{1}{2}u^2 \Big|^t$$

$$= \frac{1}{2}\lim_{t\to 2^-} \left(\sqrt[3]{x^3-8}\right)^2 \Big|_0^t = \frac{1}{2}\lim_{t\to 2^-}\left[\left(\sqrt[3]{t^3-8}\right)^2 - 4\right]$$

$$= \frac{1}{2}(-4) = \boxed{-2}$$ "Convergent"

b) $\int_{2/3}^1 \frac{1}{\sqrt[3]{3x-2}}\,dx = \lim_{t\to \frac{2}{3}^+} \int_t^1 \frac{1}{\sqrt[3]{3x-2}}\,dx$

$$\begin{cases} \text{Let } u = \sqrt[3]{3x-2} \\ u^3 = 3x-2 \\ 3u^2\,du = 3\,dx \\ u^2\,du = dx. \end{cases}$$



$$= \lim_{t\to \frac{2}{3}^+} \int \frac{1}{\cancel{u}} \cdot u^{\cancel{2}}\,du.$$

$$= \lim_{t\to \frac{2}{3}^+} \frac{u^2}{2} = \frac{1}{2}\lim_{t\to \frac{2}{3}^+} \left(\sqrt[3]{3x-2}\right)^2 \Big|_t^1$$

$$= \frac{1}{2}\lim_{t\to \frac{2}{3}^+}\left[1 - \left(\sqrt[3]{3t-2}\right)^2\right] = \frac{1}{2}(1) = \frac{1}{2}$$

"Convergent"

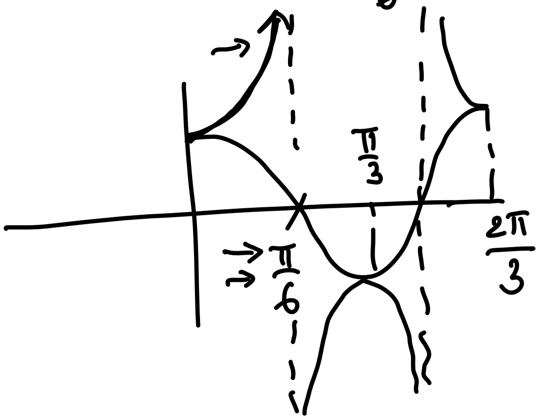
$\int \tan x\,dx = \ln|\sec x| + C.$

c) $\int_0^{\frac{\pi}{6}} \tan(3x)\,dx = \lim_{t \to \frac{\pi}{6}^-} \int_0^t \tan(3x)\,dx = \lim_{t \to \frac{\pi}{6}^-} \frac{\ln|\sec(3x)|}{3} \Big|_0^t$

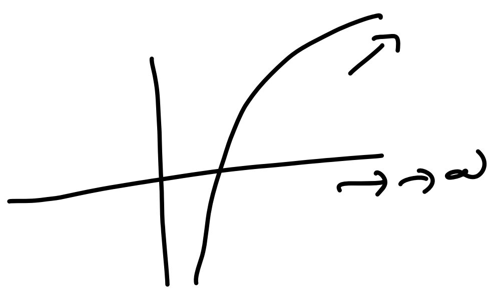
$0 \quad t \to \frac{\pi}{6}$

$$= \frac{1}{3} \lim_{t \to \frac{\pi}{6}^-} \Big[\ln|\sec(3t)| - \underbrace{\ln 1}_{=0} \Big] = \infty$$

"divergent"



$\ln|\infty|$



d) $\int_3^6 \frac{x}{x^2 - 25}\,dx$

Ex: Test for convergence / divergence.

a) $\int_{2>0}^{\infty} \frac{\sqrt[3]{x^8}}{x^4}\,dx = \int_2^{\infty} \frac{1}{x^p}\,dx$

$$= \int_2^{\infty} \frac{x^{8/3}}{x^4}\,dx = \int_2^{\infty} \frac{1}{x^{4-\frac{8}{3}}}\,dx = \int_2^{\infty} \frac{1}{x^{4/3 > 1}}\,dx$$

$p = \frac{4}{3} > 1 \Rightarrow$ By p-Test it's convergent.

Comparison test for improper integrals: (C.T.T. : comparison test Theorem)

Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $a \ge 0$

a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

$\int f(x)dx \ge \int g(x)dx$

$\int$(bigger) is convergent $\Rightarrow$ $\int$(smaller) is also convergent.

To prove $\int f(x)dx$ to be convergent
$\Rightarrow$ Construct a "bigger"

∞

b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

$\int f(x)dx \ge \int g(x)dx$

If $\int$(smaller) is divergent $\Longrightarrow$ $\int$(bigger) is also divergent.

To prove $\int f(x)dx$ to be divergent
$\Rightarrow$ Construct "smaller"

Ex: Test for convergence / divergence:

a) $\int_0^\infty e^{-x^2}\,dx$

(b) $\displaystyle\int_1^\infty \frac{7x^2+5x+2}{4x^7+3x^2+1}\,dx \le \int_1^\infty \frac{7x^2+5x^2+2x^2}{x^7}\,dx = \int_1^\infty \frac{14x^2}{x^7}\,dx = 14\int_1^\infty \frac{1}{x^5}\,dx$ $\left\{\begin{array}{l} p=5>1 \\ \text{is convergent} \\ \text{by } p\text{-Test.}\end{array}\right.$

Dominant terms: $\dfrac{x^2}{x^7} = \dfrac{1}{x^5} > 1$

By C.T.T. $\Rightarrow \displaystyle\int_1^\infty f(x)\,dx$ is also convergent.

$$\frac{a}{B} \le \frac{a}{b} \le \frac{A}{b}$$

Given

convergent

divergent

(c) $\displaystyle\int_2^\infty \frac{4x^3+3x+2\cdot 1}{\sqrt{\cancel{7}x^9+\cancel{x^8}+\cancel{3}}}\,dx \le \int_2^\infty \frac{4x^3+3x^3+2x^3}{\sqrt{x^9}}\,dx = \int_2^\infty \frac{9x^3}{\sqrt{x^9}}\,dx = 9\int_2^\infty \frac{1}{x^{\frac{9}{2}-3}}\,dx$

Dominant terms: $\dfrac{x^3}{\sqrt{x^9}} = \dfrac{1}{x^{\frac{9}{2}-3}} = \dfrac{1}{x^{3/2}} > 1$

$\displaystyle = 9\int_2^\infty \frac{1}{x^{3/2}}\,dx$ $\left\{\begin{array}{l} p=\frac{3}{2}>1 \\ \text{is convergent} \\ \text{by } p\text{-Test.}\end{array}\right.$

By C.T.T. $\displaystyle\int_2^\infty f(x)\,dx$ is also convergent

$\displaystyle\int_5^\infty \frac{2x^2+7x+3}{\sqrt[3]{8x^8+5x^4+7\cdot 1}}\,dx \ge \int_5^\infty \frac{x^2}{\sqrt[3]{8x^8+5x^8+7x^8}}\,dx = \int_5^\infty \frac{x^2}{\sqrt[3]{20x^8}}\,dx$

$\displaystyle \frac{1}{\sqrt[3]{20}}\int_5^\infty \frac{x^2}{\sqrt[3]{x^8}}\,dx = \frac{1}{\sqrt[3]{20}}\int_5^\infty \frac{1}{x^{\frac{8}{3}-2}}\,dx = \frac{1}{\sqrt[3]{20}}\int_5^\infty \frac{1}{x^{2/3}}\,dx$

Dominant terms: $\dfrac{x^2}{\sqrt[3]{x^8}} = \dfrac{1}{x^{\frac{8}{3}-2}} = \dfrac{1}{x^{2/3}} < 1$

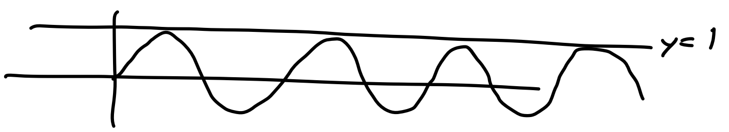
$p = \frac{2}{3} < 1 \Rightarrow$ divergent by p-Test.

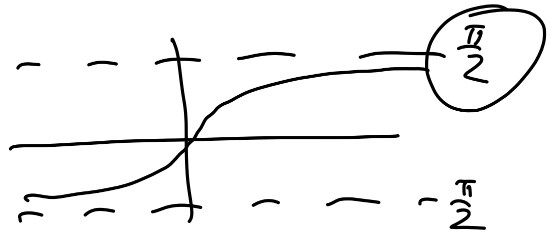
$\therefore$ by C.T.T. $\Rightarrow \displaystyle\int_5^\infty f(x)\,dx$ is also divergent.

d) $\displaystyle\int_1^{\infty}\frac{1+e^{-x}}{x}\,dx \geq \int_1^{\infty}\frac{1}{x}\,dx$ $\left\{\begin{array}{l} p=1 \text{ is divergent} \\ \text{by P-Test.}\end{array}\right.$

Dominant terms: $\frac{1}{x^1} \Rightarrow p=1 \Rightarrow$ div. $\Rightarrow$ construct "smaller"

$\therefore$ by C.T.T. $\displaystyle\int_1^{\infty} f(x)\,dx$ is also divergent.





e) $\displaystyle\int_1^{\infty}\frac{\sin(5x-3)+4\tan^{-1}(3x)}{x^3+3x+4}\,dx \leq \int_1^{\infty}\frac{\pi+2\pi}{x^3}\,dx = 3\pi\int_1^{\infty}\frac{1}{x^3}\,dx$ $\left\{\begin{array}{l} p=3>1 \text{ is} \\ \text{convergent} \\ \text{by P-Test.}\end{array}\right.$

Dominant terms: $\dfrac{1+4\cdot\frac{\pi}{2}}{x^3} = \dfrac{1+2\pi}{x^3}$ $\leftarrow$ $p=3>1 \Rightarrow$ Convergent $\Rightarrow$ construct "bigger"

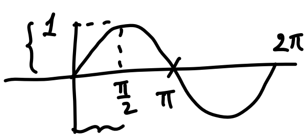
$\therefore$ By C.T.T. $\Rightarrow$ $\displaystyle\int_1^{\infty} f(x)\,dx$ is also convergent.

f) $\int_1^{\infty} \frac{e^{x^2+x+1}}{1} dx \geq \int_1^{\infty} e\, dx = e \int_1^{\infty} 1\, dx = e \int_1^{\infty} \frac{1}{x^0} dx \Rightarrow p = 0 < 1 \Rightarrow$ divergent by p-Test.

$\therefore$ by C.T.T. $\int_1^{\infty} e^{x^2+x+1} dx$ is also divergent.

p-Test $\int_{a>0}^{\infty} \frac{1}{x^p} dx$

☆ $\int_0^{\pi/2} \frac{1}{x \sin(x)} dx$

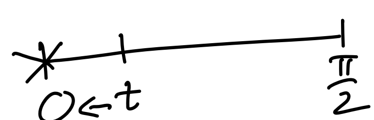


for $0 \leq x \leq \frac{\pi}{2} \Rightarrow 0 \leq \sin(x) \leq 1$.

multiply by x: $0 \leq x \sin x \leq x$.

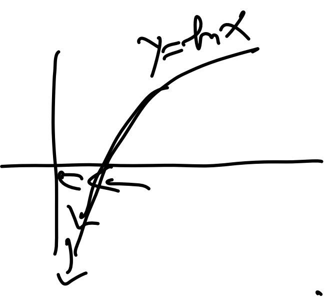
Flip: $\frac{1}{x \sin x} \geq \frac{1}{x}$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{x \sin x} dx \geq \int_0^{\frac{\pi}{2}} \frac{1}{x} dx$. ← Is this p-Test? NOT.



$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{x} dx = \lim_{t \to 0^+} \int_t^{\frac{\pi}{2}} \frac{1}{x} dx = \lim_{t \to 0^+} \ln|x| \Big|_t^{\frac{\pi}{2}}$

$= \lim_{t \to 0^+} \left[\ln \frac{\pi}{2} - \ln|t| \right] = \infty \Rightarrow$ divergent



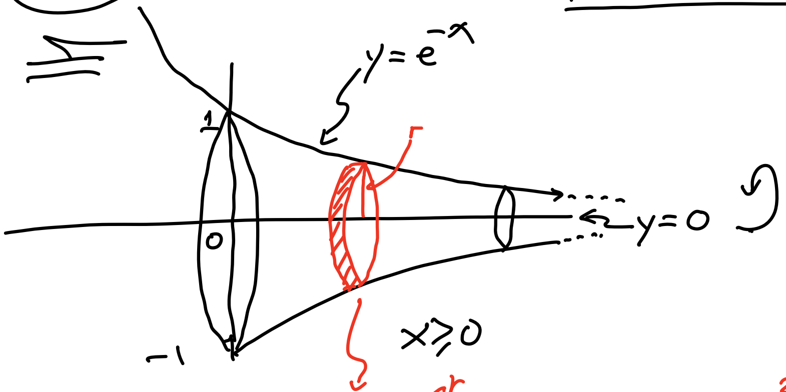
$\therefore$ by C.T.T. $\int_0^{\frac{\pi}{2}} \frac{1}{x \sin x} dx$ is also divergent. ??

$-\infty$

Try: $\int_0 \frac{dx}{x^5 \cos(2x)}$ . . .

Ex: Find the volume of the following:

a) The region bounded by $\boxed{y = e^{-x}; \;\; y = 0; \;\; for \;\; x \ge 0}$ is rotated about the x – axis.



$dx \Rightarrow V = \pi r^2 dx$ where r is $\left.\begin{array}{l} y_{top} = e^{-x} \\ y_{bot} = 0 \end{array}\right\} = e^{-x} - 0 = e^{-x}$.

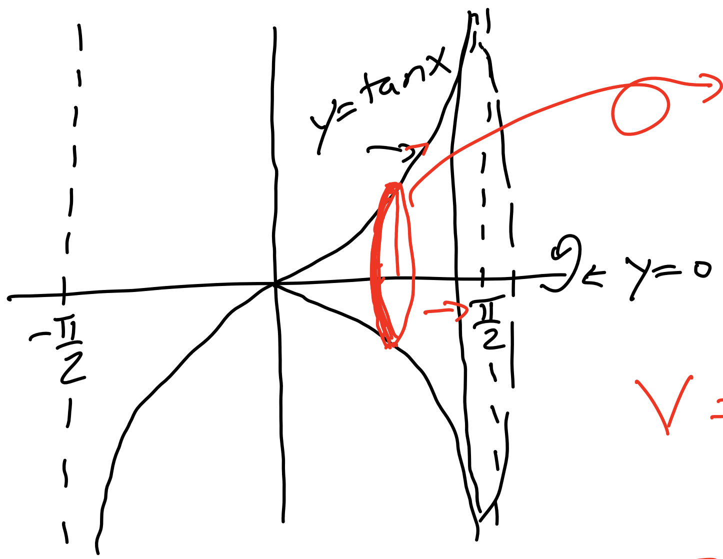
$$\Rightarrow \quad V = \int_0^{\infty} \pi \left(e^{-x}\right)^2 dx = \lim_{t \to \infty} \int_0^t \pi e^{-2x} dx = \pi \lim_{t \to \infty} \left. \frac{e^{-2x}}{-2} \right|_0^t$$

$$= -\frac{\pi}{2} \lim_{t \to \infty} \left(\underbrace{e^{-2t}}_{\to 0} - e^0\right) = \frac{\pi}{2}$$

Convergent

$0 \qquad t \to \infty$

b) The region bounded by $y = \tan(x)$; $y = 0$ *for* $0 \le x \le \frac{\pi}{2}$ is rotated about the x – axis.



r

dx

$V = \pi \cdot r^2 \cdot dx$

where r is $\left.\begin{array}{l} y_{top} = \tan x \\ y_{bot} = 0 \end{array}\right\} = \tan x - 0 = \tan x$.

$$V = \pi \int_0^{\frac{\pi}{2}} (\tan x)^2 dx$$

$$= \pi \lim_{t \to \frac{\pi}{2}^-} \int_0^t \left(\sec^2 x - 1\right) dx$$

$$= \pi \lim_{t \to \frac{\pi}{2}^-} \Big[\tan x - x\Big]_0^t$$

$$= \pi \lim_{t \to \frac{\pi}{2}^-} \Big[\tan t - t\Big] = \infty \leftarrow \text{divergent.}$$

$0 \qquad t \to \frac{\pi}{2}$

$\frac{\pi}{2}$

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