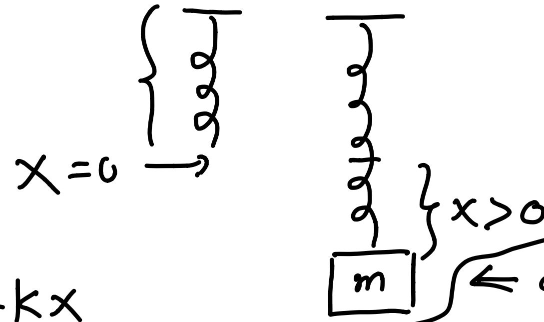
Chapter 5 ***Spring / Mass Systems: Oscillations of a Mechanical System***

Spring/Mass Systems:

Free Undamped Motion.



← Equilibrium pt.

$F_s = F_g \Rightarrow kx = mg$

Hooke's Law: $F = -kx$

Newton's Second Law: $F = \underline{m}a = -kx$

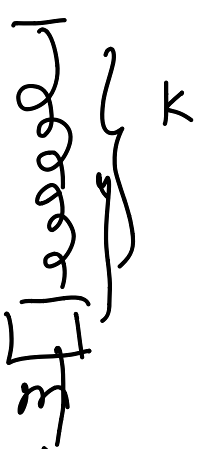
$$m \cdot \frac{d^2x}{dt^2} = -kx. \Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x \Rightarrow \underline{\frac{d^2x}{dt} + \frac{k}{m}x = 0}$$

DE of Free Undamped Motion: (Harmonic motion)

$$\frac{d^2x}{dt^2} + \omega^2 \cdot y = 0 \quad \text{where } \omega^2 = \frac{k}{m}.$$

$$P(\lambda) = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm \omega i.$$

$$\underline{x(t) = A\cos(\omega t) + B\sin(\omega t)} \quad \boxed{\begin{aligned} x(0) &= x_0 \\ x'(0) &= V_0 \end{aligned}}$$

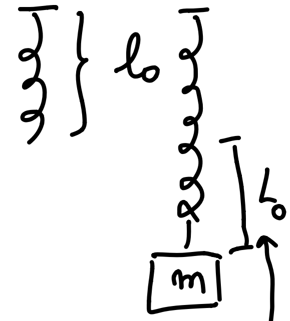


Mathematical Formulation

Statement of the problem: A mass of m kilograms is attached to the end of a spring whose natural length is l_0. At t=0, the mass is displaced a distance L_0 meters from its equilibrium position and released with a velocity v_0 meters/second. We wish to determine the IVP that governs the resulting motion.

$x(t)$

Hook's Law: $F_s = -kx$



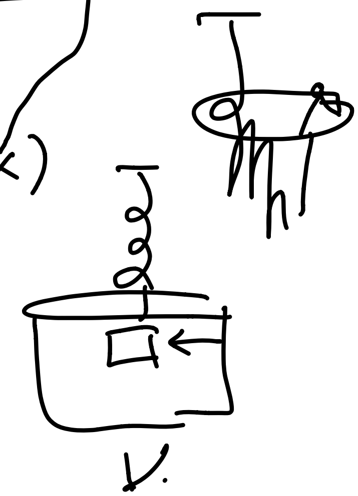
At equilibrium position: $F_g = F_s$

$mg = -kx = -kL_0$

$mg + kL_0 = 0$

At time t = 0.

$F_s = -k(x)$



In motion, we have the following forces acting on the mass.

1. $F_g = mg$

$F_s = -k(\text{dist})$

2. $F_s = -k[L_0 + y(t)]$ Where $y(t)$ is the displacement from its equilibrium position at time t.

3. A damping force F_d. In general, the motion will be damped due, for example, to air resistance, or an external damping system, such as a dashpot. We assume that any damping forces that are present are directly proportional to the velocity of the mass. $F_d = -c\boxed{\frac{dy}{dt}}$

4. Any external driving forces $F(t)$ that are present. For example, the top of the spring or the mass itself may be subjected to an external force.

So the total force acting on the system will be the sum of the preceding forces. Thus, using Newton's second law, the DE governing the motion of the mass is

$$F = ma = m\frac{d^2y}{dt^2} = F_g + F_s + F_d + F(t) = mg - k(L_0 + y) - c\frac{dy}{dt} + F(t)$$

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{1}{m}F(t)$$ with the initial condition $y(0) = y_0; y'(0) = v_0$

$\div m$

$$m\frac{d^2y}{dt^2} = \underbrace{(mg - kL_0)}_{=0} - ky - c\frac{dy}{dt} + F(t)$$

$$\to m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)$$

Free Oscillations of a Mechanical System (No external force)

We first consider the case when there are no external forces acting on the system, and then we have the following homogeneous DE. $\begin{cases} y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \\ y(0) = y_0;\, y'(0) = v_0 \end{cases}$.

Case 1: ***Simple Harmonic Motion***: There is no damping i.e. when c = 0.

$$\begin{cases} y'' + \boxed{\frac{c}{m}y} + \frac{k}{m}y = 0 \Rightarrow y'' + \frac{k}{m}y = 0;\ \ let\ \varpi^2 = \frac{k}{m} \Rightarrow y'' + \varpi^2 y = 0 \\ subject\ \ to\ \ y(0) = y_0; \quad y'(0) = v_0 \end{cases}$$

Sol: The characteristic polynomial: $y'' + \varpi^2 y = 0$

$$p(\lambda) = \lambda^2 + \varpi^2 = 0 \Rightarrow \lambda = \pm\varpi i \Rightarrow y(t) = C_1\cos(\varpi t) + C_2\sin(\varpi t)$$

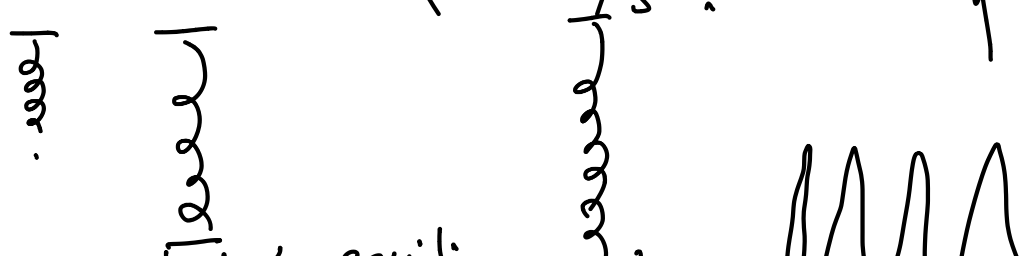
$$y(t) = A\cos(\varpi t) + B\sin(\varpi t) \Rightarrow \begin{cases} y(0) = C_1 = y_0 \\ y'(t) = -\varpi C_1\sin(\varpi t) + \varpi C_2\cos(\varpi t) \\ \Rightarrow y'(0) = \varpi C_2 = v_0 \Rightarrow C_2 = \frac{v_0}{\varpi} \end{cases}$$

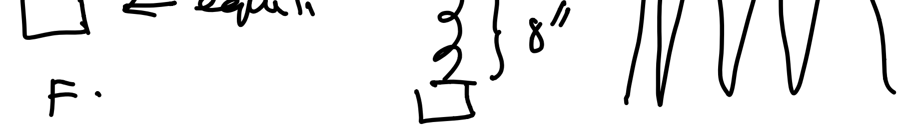
$$Solution: \boxed{y(t) = y_0\cos(\varpi t) + \frac{v_0}{\varpi}\sin(\varpi t)}$$

gravity: $g = \begin{cases} 9.8\ m/s^2. \\ 32\ ft/s^2. \end{cases}$

$\downarrow$ $y > 0$, $y' > 0$

$\uparrow$ $y' < 0$





Ex1: A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}\ ft/\sec$. Determine the equation of free motion.

Sol:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = F(t) \qquad \begin{cases} y(0) = y_0 \\ y'(0) = v_0 \end{cases}.$$

$c = 0 \to$ No damping force $\cdot$; $F(t) = 0 \Rightarrow$ No External force $\Big\}\ y'' + \frac{k}{m}y = 0.$

find k & m.

$$F_g = 2 = m \cdot g = m \cdot 32 \Rightarrow m = \frac{2}{32} = \frac{1}{16} \text{ slugs.}$$

$$F_s = ky \Rightarrow 2\,lb = k\left(\tfrac{1}{2}\,ft\right) \Rightarrow k = 4\ lb/ft.$$

$$y'' + \frac{4}{1/16}y = 0 \Rightarrow \underline{y'' + 64y = 0} \qquad \begin{cases} y(0) = \frac{8}{12} = \frac{2}{3}\ ft. \\ y'(0) = -\frac{4}{3}\ ft/sec. \end{cases}$$

Convert the solution $\boxed{y = y_0\cos(\varpi t) + \frac{v_0}{\varpi}\sin(\varpi t)}$ into Phase – Amplitude form: $y(t) = A\cos(\varpi t - \delta)$

$$p(\lambda) = \lambda^2 + 64 = 0 \Rightarrow \lambda = \pm 8i.$$

$$y(t) = A\cos(8t) + B\sin(8t)$$

$$y(0) = A = \frac{2}{3}$$

$$y'(t) = -8A\sin(8t) + 8B\cos(8t)$$

$$y'(0) = 8B = -\frac{4}{3} \Rightarrow B = -\frac{1}{6}.$$

$$\boxed{y(t) = \frac{2}{3}\cos(8t) - \frac{1}{6}\sin(8t)} \checkmark$$

$y(0)=y_0$

phase- Amplitude form.

Ex: Let $y''+y=0;\ y(0)=-4;\ y'(0)=3$

$P(\lambda)=\lambda^2+1=0 \Rightarrow \lambda=\pm i.$ $\qquad : \quad y''+\omega^2 y=0$

$y(t)=c_1\cos t+c_2\sin t = A\cos(\omega t-\delta)$

$y=A\cos(t-\delta). \Rightarrow A=\sqrt{y_0^2+\left(\frac{v_0}{\omega}\right)^2}\ ;\ \delta=\tan^{-1}\left(\frac{v_0}{\omega y_0}\right)$

$y(0)=A\cos(-\delta)=A\cos\delta=-4$

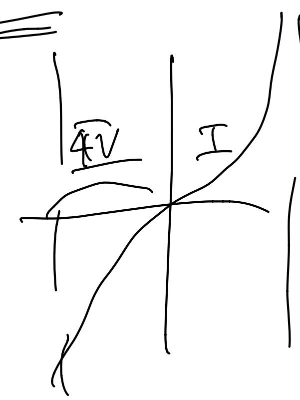
$y'(t)=-A\sin(t-\delta)$

$y'(0)=-A\sin(-\delta)=A\sin\delta=3$

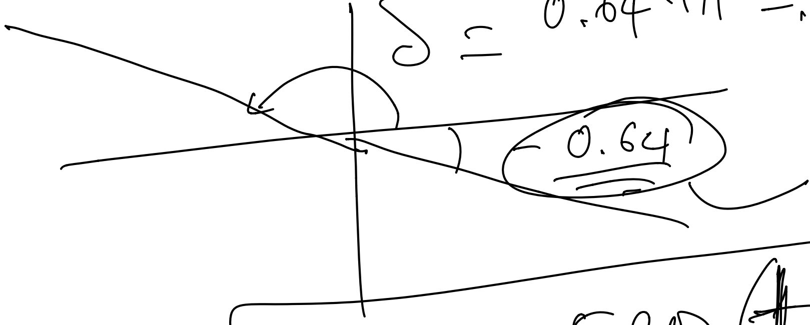
$$\begin{aligned} &A^2\cos^2\delta=16\\ +\ &A^2\sin^2\delta=9\\ \hline &A^2=25\Rightarrow A=5. \end{aligned}$$

$\Rightarrow \dfrac{\cancel{A}\sin\delta}{\cancel{A}\cos\delta}=-\dfrac{3}{4} \Rightarrow \tan\delta=-\dfrac{3}{4} \Rightarrow \delta=\tan^{-1}\left(-\dfrac{3}{4}\right)=-0.6435$

Note: $\left.\begin{aligned}A\cos\delta=-4<0\\ A\sin\delta=3>0\end{aligned}\right\}\ \delta\in Q\,II.$



$\delta=0.64+\pi=2.49.$



Sol: $\boxed{y(t)=5\cos(t-2.49)}$

Ex: Solve $y''+9y=0;\ y(0)=1;\ y'(0)=-3$

a) Find the solution in phase-amplitude form;

b) Find the time at which the mass crosses the equilibrium position for the first time.

c) Find the maximum speed of the mass.

Sol: $p(\lambda)=\lambda^2+9=0 \Rightarrow \lambda=\pm 3i \Rightarrow y(t)=A\cos(\omega t-\delta)$

$$y(t)=A\cos(3t-\delta).$$

$$y(0)=A\cos(-\delta)=A\cos\delta=1.$$

$$y'(t)=-A\cdot 3\cdot\sin(3t-\delta).$$

$$y'(0)=-3A\sin(-\delta)=\boxed{3A\sin\delta=-3.}$$

$$\begin{aligned}A\cos\delta&=1\\ A\sin\delta&=-1\end{aligned} \Rightarrow \begin{cases}A^2\cos^2\delta=1\\ +\ A^2\sin^2\delta=1\end{cases}$$

$$A^2=2 \Rightarrow A=\sqrt{2}.$$

$$\frac{3\cancel{A}\sin\delta}{\cancel{A}\cos\delta}=-3 \Rightarrow \tan\delta=-1 \Rightarrow \delta=\begin{cases}\frac{\pi}{4}\\ -\frac{\pi}{4}\end{cases}$$

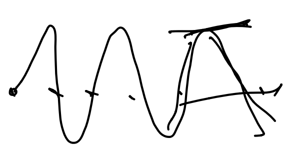
$$\boxed{y(t)=\sqrt{2}\cos\left(3t+\frac{\pi}{4}\right)}.$$

b) First time, it crosses the equilibrium pts.

$$y(t)=\sqrt{2}\cos\left(3t+\frac{\pi}{4}\right)\overset{?}{=}0.$$

$$3t+\frac{\pi}{4}=\frac{\pi}{2}$$

$$3t=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}. \Rightarrow t=\frac{\pi}{12}$$



$$|y'(t)|=\left|-3\sqrt{2}\sin\left(3t+\frac{\pi}{4}\right)\right|$$

max. speed, $|y(t)| = \Big|$ $\Big|_{t=\frac{\pi}{12}}$.

$= 0.98m$

Ex: A mass of 1 kg m stretches a spring 9.8 cm. Let y =0 denote the equilibrium position of the mass after ~~it is~~ attached to the spring. Suppose that the spring acts linearly if it is not stretched or compressed more than 1 m from its length before the mass is attached. If the spring stretches more than this amount, then it no longer obeys Hooke's law (that is, the spring is deformed if it stretches too much). Ignore friction and air resistance. Consider the following questions:

1. Use Newton's laws to derive a differential equation that describes the motion.
2. What is the period of the motion?
3. The spring is stretched to an initial position $y(0) = y_0$ and released with zero initial velocity. For what values of y_0 will the spring not be stretch so much that it deforms?
4. The spring is pushed from the equilibrium position with initial velocity v_0. For what values of v_0 will the spring not be damaged?

Solution:

1. The gravitational force on the mass (that is, its weight) is $mg = 9.8N$, which exactly balances the restoring force of the spring when the mass is at rest at the equilibrium position. The mass stretches the spring by 0.098m, so the spring constant is $k = 9.8N/0.098m = 100N/m$. Friction is negligible, so the restoring force of he spring equals the weight ~~of he mass~~. Therefore,
$my'' = -F_{restoring} = -ky = -100y \Leftrightarrow y'' + 100y = 0$

2. The general solution is $y(t) = c_1 \cos 10t + c_2 \sin 10t$, thus the period of the motion is $2\pi/10\,\text{sec}$.

3. We substitute the initial values, we have:
$$\begin{aligned} y(0) &= C_1 = y_0 \\ y'(0) &= 10C_2 = 0 \Rightarrow C_2 = 0 \end{aligned} \Rightarrow y(t) = y_0 \cos(10t)$$
Hence, the maximum displacement of the mass is y_0 meters from the equilibrium. To avoid damaging the spring, we must not stretch it more than 1 m from its initial length before the mass is attached. The mass stretches the spring when it is attached; in this case, the equilibrium position $y = 0$ corresponds to an elongation of 0.098m. Thus the spring is not damage if $|y_0| < 1 - 0.098 = 0.902m$

4. The spring is pushed from the equilibrium point: $y(0) = 0;\ y'(0) = v_0$
$$y(t) = C_1 \cos 10t + C_2 \sin 10t \Rightarrow \begin{cases} y(0) = C_1 = 0 \\ y'(0) = 10C_2 = v_0 \Rightarrow C_2 = \dfrac{v_0}{10} \end{cases}$$
Solution: $y(t) = \dfrac{v_0}{10} \sin(10t)$
The initial conditions imply $c_1 = 0;\ c_2 = v_0/10$. The maximum amplitude of the spring is $v_0/10\,\text{m}$. The spring is undamaged if $|v_0/10| < 1 - 0.098$, that is $|v_0| < 9.02m/s$

Damping:

$$\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=0;\ \ y(0)=y_0;\ y'(0)=v_0$$

$c \neq 0$

$$p(\lambda)=\lambda^2+\frac{c}{m}\lambda+\frac{k}{m}=0 \Rightarrow \lambda=\frac{-\frac{c}{m}\pm\frac{1}{m}\sqrt{c^2-4mk}}{2}=\frac{-c\pm\sqrt{c^2-4mk}}{2m}$$

> 0

a) Overdamped: $c^2-4km>0 \Leftrightarrow \frac{c^2}{4km}>1 \Rightarrow$ (Two distinct real roots)

b) Critically damped if $c^2-4km=0 \Leftrightarrow \frac{c^2}{4km}=1 \Rightarrow$ (Repeated real root)

c) Underdamped if $c^2-4km<0 \Leftrightarrow \frac{c^2}{4km}<1 \Rightarrow$ (Two complex conjugate roots)

Overdamped: When we have two distinct roots, say λ_1 *and* λ_2, then we clearly see that the complement solutions of the DE, is $y(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$

Ex: A mass spring obey the DE: $\boxed{\frac{d^2y}{dt^2}+5\frac{dy}{dt}+4y=0;}\ y(0)=1;\ y'(0)=1.$

a) Determine when the mass reaches its extremum and find its extreme value(s).
b) Does the mass cross the equilibrium position.

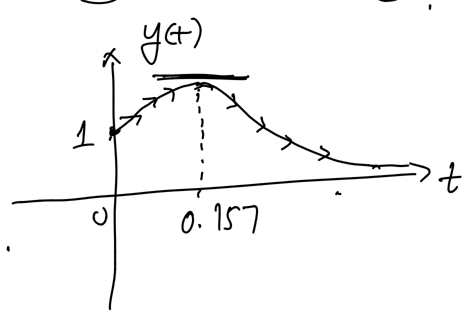
$p(\lambda)=\lambda^2+5\lambda+4=0 \Rightarrow (\lambda+1)(\lambda+4)=0 \Rightarrow \lambda=-1,-4$ {overdamped}

$y(t)=C_1e^{-t}+C_2e^{-4t}.$

$$\begin{cases} y(0)=C_1+C_2=1 \\ y'(0)=-C_1-4C_2=1 \end{cases}$$

$-3C_2=2 \Rightarrow C_2=-\frac{2}{3} \Rightarrow C_1=1+\frac{2}{3}=\frac{5}{3}.$

$y(t)=\frac{5}{3}e^{-t}-\frac{2}{3}e^{-4t}$



Extreme-Value $\Rightarrow$ $y'=-\frac{5}{3}e^{-t}+\frac{8}{3}e^{-4t}=0.$

$-5e^{-t}+8e^{-4t}=0$

$\frac{8e^{-4t}}{5e^{-4t}}=\frac{5e^{-t}}{5e^{-4t}}=e^{3t}=\frac{8}{5}$

$t=\frac{1}{3}\ln\left(\frac{8}{5}\right)=0.157$ sec.

Ex: Let the motion of a linear pendulum be governed by the equation: $y''+4y'+3y=0$;

a) Suppose the pendulum initially is at the equilibrium position, that is, $y(0)=0$, and that $y'(0)=v_0 \neq 0$. Does the pendulum ever cross the equilibrium? Explain why or why not.

$$p(\lambda)=\lambda^2+4\lambda+3=0 \Rightarrow (\lambda+1)(\lambda+3)=0 \quad \lambda=-1,-3$$

$$y(t)=c_1e^{-t}+c_2e^{-3t} \quad \begin{cases} y(0)=0 \\ y'(0)=V_0 \end{cases} \Rightarrow \begin{cases} y(0)=c_1+c_2=0 \\ y'(0)=-c_1-3c_2=V_0 \end{cases}$$

$$-2c_2=V_0 \Rightarrow c_2=-\tfrac{1}{2}V_0$$

$$c_1=\tfrac{1}{2}V_0.$$

$$y(t)=\tfrac{1}{2}V_0e^{-t}-\tfrac{1}{2}V_0e^{-3t}$$

b) Suppose $y(0)=1$. Show that the pendulum crosses the equilibrium once if $v_0<-3$

Cross the equilibrium $\Rightarrow$ $y(t)=0$ for t.

$$y(t)=\tfrac{1}{2}\underbrace{V_0}_{\neq 0}\left(e^{-t}-e^{-3t}\right)=0 \Rightarrow e^{-t}-e^{-3t}=0 \Rightarrow \frac{e^{-t}}{e^{-3t}}=\frac{e^{-3t}}{e^{-3t}}.$$

$$e^{2t}=1 \Rightarrow t=0$$

$$y(t)=c_1e^{-t}+c_2e^{-3t} \quad \begin{cases} y(0)=1 \\ y'(0)=V_0<-3. \end{cases}$$

$$\begin{cases} y(0)=c_1+c_2=1 \\ y'(0)=-c_1-3c_2=V_0 \end{cases} \quad -2c_2=1+V_0, \quad c_2=-\frac{1+V_0}{2}, \qquad c_1=1+\frac{1+V_0}{2}=\frac{3+V_0}{2}$$

$$Y=\frac{3+V_0}{2}e^{-t}-\frac{1+V_0}{2}e^{-3t}.$$

c) Are there any initial conditions for which the pendulum crosses the equilibrium position exactly twice. Explain why or why not?

Cross equilibrium pts $\Rightarrow$ $Y(t)=0 \Rightarrow \boxed{\frac{3+V_0}{2}\cdot e^{-t}-\frac{1+V_0}{2}e^{-3t}}=0.$

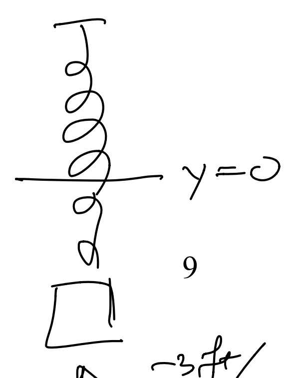
$$(3+V_0)e^{-t}=(1+V_0)e^{-3t} \Rightarrow \frac{e^{-t}}{e^{-3t}}=\frac{1+V_0}{3+V_0} \Rightarrow \boxed{e^{2t}=\frac{1+V_0}{3+V_0}} \Rightarrow t=\frac{1}{2}\ln\left(\frac{1+V_0}{3+V_0}\right)$$

d) Find the solution satisfying the initial condition $y(0)=x_0$; $y'(0)=v_0$

$$\left(\frac{1+V_0}{3+V_0}\right)>1 \Rightarrow \frac{1+V_0}{3+V_0}-1>0.$$

$$\frac{1+V_0-3-V_0}{3+V_0}>0 \Rightarrow \frac{-2}{3+V_0}>0.$$

$$\Rightarrow 3+V_0<0 \Rightarrow \boxed{V_0<-3}$$



Critically Damped Motion

$$\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=0;\ y(0)=y_0;\ y'(0)=v_0$$

$$p(\lambda)=\lambda^2+\frac{c}{m}\lambda+\frac{k}{m}=0\Rightarrow\lambda=\frac{-c\pm\sqrt{c^2-4mk}}{2m}\Rightarrow c^2-4mk=0;\ and\ \lambda=-\frac{c}{2m},-\frac{c}{2m}$$

$$y(t)=e^{-\frac{c}{2m}t}(C_1+C_2t)$$

Ex1: An 8 – pound weight stretches a spring 2 ft. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the weight is released from the equilibrium position with an upward velocity of 3 ft/sec.

* force: $8\,lb = m\cdot g \Rightarrow m = \frac{8}{g} = \frac{8}{32} = \frac{1}{4}$ slug.

* force $= 8\,lb = kx \Rightarrow k = \frac{8}{x} = \frac{8}{2} = 4$.

$y'' + \frac{c}{m}y' + \frac{k}{m}\cdot y = 0$; (2 times y') $\Rightarrow c=2$.

$y'' + \frac{2}{1/4}y' + \frac{4}{1/4}y = 0$

$\boxed{y''+8y'+16y=0. \quad \begin{cases} y(0)=0 \\ y'(0)=-3 \end{cases}}$

$p(\lambda) = \lambda^2+8\lambda+16 = (\lambda+4)^2=0 \Rightarrow \lambda=-4,-4$

$y(t) = e^{-4t}(C_1+C_2t)$.

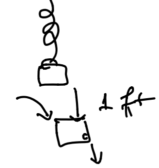
$y(0) = C_1 = 0$

$y'(t) = e^{-4t}[-4C_1 - 4C_2t + C_2]$.

$y'(0) = C_2 = -3$.

$y(t) = e^{-4t}(0-3t) = -3te^{-4t}$

Ex: Consider the IVP: $y''+y'+\frac{1}{4}x=0;\ y(0)=1,\ y'(0)=1.$



$$p(\lambda) = \lambda^2+\lambda+\frac{1}{4}=0 \Rightarrow \left(\lambda+\frac{1}{2}\right)^2=0 \Rightarrow \lambda=-\frac{1}{2},\ -\frac{1}{2}$$

$$y(t)=e^{-\frac{1}{2}t}\left(c_1+c_2t\right) \Rightarrow y(0)=c_1=1.$$

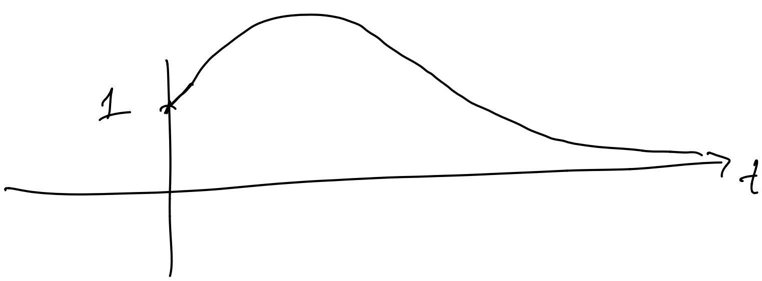
$$y'(t)=e^{-\frac{1}{2}t}\left(-\frac{1}{2}c_1-\frac{1}{2}c_2t+c_2\right) \Rightarrow y'(0)=-\frac{1}{2}(1)+c_2=1.$$

$$c_2=1+\frac{1}{2}=\frac{3}{2}.$$

$$y(t)=e^{-\frac{1}{2}t}\left[1+\frac{3}{2}t\right].$$

Does the mass ever cross the equilibrium pt.

$$y(t)=e^{-\frac{1}{2}t}\underbrace{\left[1+\frac{3}{2}t\right]}=0 \Rightarrow t=-\frac{2}{3} \Rightarrow$$ Never cross the equilibrium pt,



Note: The motion of a critically damped pendulum is similar to that of an Overdamped pendulum. In particular.

1. The bob does not oscillate around the equilibrium (x = 0) position;
2. As $t \to \infty$, the bob tends to the equilibrium position; and
3. The bob crosses the equilibrium position at most once.

It is difficult in practice to achieve critical damping, because the relation $c^2 = 4m^2gl$ must be satisfied precisely. Even if we assume that the frictional forces are perfectly proportional to the current angular velocity of the pendulum, any error in measuring the mass or the length of the pendulum means that we are likely to be in the Overdamped or underdamped case.

Underdamped:

$$\frac{d^2x}{dt^2}+\frac{c}{m}\frac{dx}{dt}+\frac{k}{m}x=0;\ \ y(0)=y_0;\ y'(0)=v_0$$

$$p(\lambda)=\lambda^2+\frac{c}{m}\lambda+\frac{k}{m}=0\Rightarrow\lambda=\frac{-\frac{c}{m}\pm\frac{1}{m}\sqrt{c^2-4mk}}{2}=\frac{-c\pm\sqrt{c^2-4mk}}{2m};\ c^2-4mk<0$$

<0

$$\Rightarrow\lambda=a\pm bi\Rightarrow y(t)=e^{at}\left(C_1\cos(bt)+C_2\sin(bt)\right)$$

Ex: Solve $9y''+30y'+29y=0;\ y(0)=1;\ y'(0)=-3$

$p(\lambda)=9\lambda^2+30\lambda+29=0$.

$\underbrace{(3\lambda)^2+2\cdot(3\lambda)\cdot 5+25}+4=0$

$(3\lambda+5)^2=-4$

$3\lambda+5=\pm 2i$.

$\lambda=-\frac{5}{3}\pm\frac{2}{3}i$.

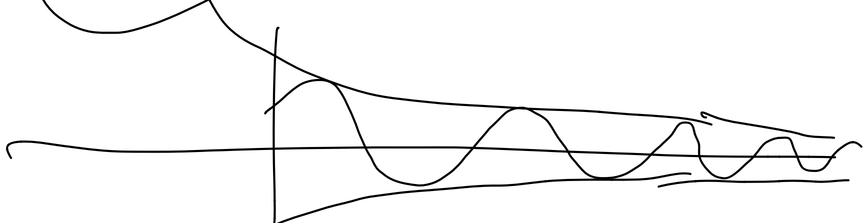
$y(t)=e^{-\frac{5}{3}t}\left[C_1\cos\left(\frac{2}{3}t\right)+C_2\sin\left(\frac{2}{3}t\right)\right]$

$y(0)=C_1=1$; $y'(t)=e^{-\frac{5}{3}t}\left[-\frac{5}{3}C_1\cos\left(\frac{2}{3}t\right)-\frac{5}{3}C_2\sin\left(\frac{2}{3}t\right)-\frac{2}{3}C_1\sin\left(\frac{2}{3}t\right)+\frac{2}{3}C_2\cos\left(\frac{2}{3}t\right)\right]$

$y'(0)=-\frac{5}{3}C_1+\frac{2}{3}C_2=-3\Rightarrow C_2=\frac{3}{2}\left(-3+\frac{5}{3}\right)=\frac{3}{2}\left(-\frac{4}{3}\right)=-2$.

$$y(t)=e^{-\frac{5}{3}t}\underbrace{\left[\cos\left(\frac{2}{3}t\right)-2\sin\left(\frac{2}{3}t\right)\right]}_{\parallel\ A\cos(\omega t-\delta)}$$

$$e^{-\frac{5}{3}t}\cdot A\cos(\omega t-\delta)$$



The Phase-Amplitude Formulation of Underdamped Solutions

We know that the solution for an underdamped is

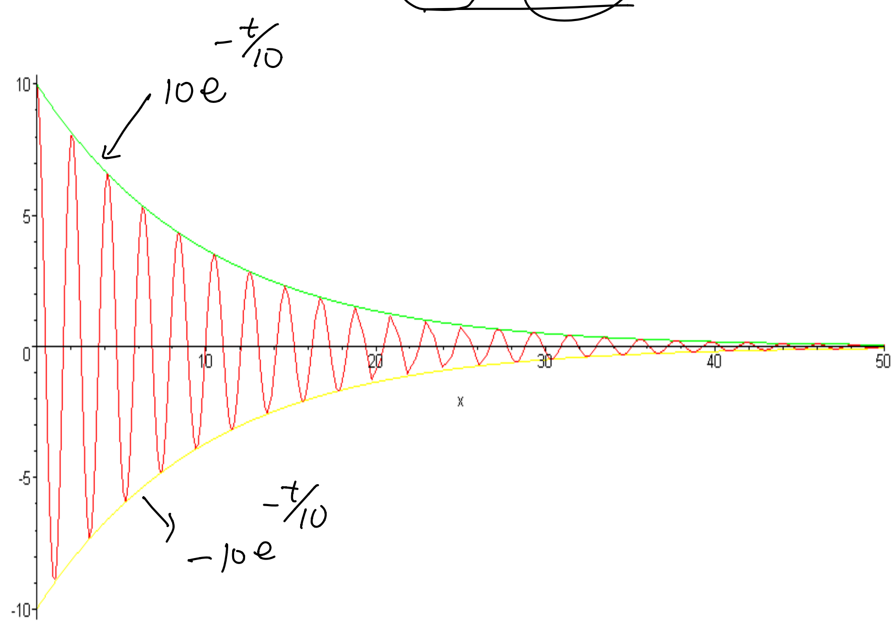
$\lambda = a \pm bi \Rightarrow y(t) = e^{at}\left(C_1 \cos(bt) + C_2 \sin(bt)\right)$.

We want to rewrite $y(t) = Ae^{at}\cos(bt - \delta)$ where b *and* δ can be computed from the initial conditions as before.

- If $a \neq 0$ then the equation is not periodic. Although the cosine function is periodic with period $2\pi/b$, so we have that
- $y\left(t + \dfrac{2\pi}{b}\right) = Ae^{a(t+2\pi/b)}\cos\left(b\left(t + \dfrac{2\pi}{b}\right) - \delta\right) = Ae^{at}e^{2a\pi/b}\cos(bt - \delta) = e^{2a\pi/b}y(t)$

  which is not $y(t)$; *if* $a \neq 0$
- We say that the equation with $a \neq 0$ is pseudoperiodic, because it behaves like a periodic function except that its amplitude is not a constant. The pseudoperiodic is $2\pi/b$, the period of the cosine term.
- Equation may be regarded as a cosine function with an exponentially decaying amplitude when $a < 0$. The term Ae^{at} is called the envelope. The graph of Ae^{at} and the graph of $-Ae^{at}$ enclose the graph of $y(t)$
- Equation allows you to determine by inspection when the mass crosses the equilibrium; you need only determine the time t for which $at - \delta$ is odd multiple of $\pi/2$

Ex: Suppose the oscillating curve of $x(t) = 10e^{-t/10}\cos(3t)$



$$my'' + cy' + ky = 0$$

Ex1: Suppose a mass of 80 kg is attached to a linear spring whose spring constant is 25N/m. If the force of friction is proportional to the current velocity of the mass with a proportionality constant of 40 kg/s, then the differential equation governing the motion is $80y'' + 40y' + 25y = 0$, the mass starts 1m below its equilibrium position with a downward initial velocity of 3m/sec. Determine the function describes the motion of the mass.

$$80y'' + 40y' + 25y = 0 \;; \quad y(0) = 1 \;, \; y'(0) = 3.$$

$$p(\lambda) = \frac{80}{5}\lambda^2 + \frac{40}{5}\lambda + \frac{25}{5} = 0$$

$$16\lambda^2 + 8\lambda + 5 = 0.$$

$$\underbrace{(4\lambda)^2 + 2(4\lambda)\cdot 1 + 1} + 4 = 0.$$

$$(4\lambda + 1)^2 = -4$$

$$4\lambda + 1 = \pm 2i$$

$$\lambda = -\frac{1}{4} \pm \frac{1}{2}i.$$

$$y(t) = e^{-\frac{1}{4}t}\left[c_1\cos\left(\tfrac{1}{2}t\right) + c_2\sin\left(\tfrac{1}{2}t\right)\right] = Ae^{-\frac{1}{4}t}\cos\left(\tfrac{1}{2}t - \delta\right)$$

$$\boxed{y(0) = A\cos(-\delta) = A\cos\delta = 1.} \checkmark$$

$$y'(t) = Ae^{-\frac{1}{4}t}\left[-\frac{1}{4}\cos\left(\tfrac{1}{2}t - \delta\right) - \frac{1}{2}\sin\left(\tfrac{1}{2}t - \delta\right)\right].$$

$$y'(0) = A\left[-\frac{1}{4}\cos\delta - \frac{1}{2}\sin(-\delta)\right] = 3.$$

$$= -\frac{1}{4}\overbrace{A\cos\delta}^{=1} + \frac{1}{2}A\sin\delta = 3.$$

$$\frac{1}{2}A\sin\delta = 3 + \frac{1}{4} = \frac{13}{4} \Rightarrow A\sin\delta = \frac{13}{2}.$$

Ex2: A 16 – pound weight is attached to a 5 – foot – long spring. At equilibrium the spring measures 8.2 ft. If the weight is pushed up and released from the rest at a point 2 ft above the equilibrium position, find the displacements $y(t)$ if it is further known that the surrounding medium offers assistance numerically equal to half the instantaneous velocity.

$$A\cos\delta = 1 \Rightarrow A^2\cos^2\delta = 1$$

$$A\sin\delta = \frac{13}{2} \Rightarrow A^2\sin^2\delta = \frac{169}{4}$$

$$A^2 = 1 + \frac{169}{4} = \frac{173}{4}$$

$$A = \sqrt{\frac{173}{4}} = \frac{\sqrt{173}}{2} = 6.576$$

$$\frac{A\sin\delta}{A\cos\delta} = \frac{13/2}{1} \Rightarrow \tan\delta = \frac{13}{2} \Rightarrow \delta = \tan^{-1}\left(\frac{13}{2}\right) \approx 1.41$$

$$\boxed{y(t) = 6.576e^{-\frac{1}{4}t}\cos\left(\frac{1}{2}t - 1.41\right)} = 0$$

a) when does the object pass the equilibrium point the 1^{st} time.

$$\frac{1}{2}t - 1.41 = \begin{cases} \frac{\pi}{2} \\ -\frac{\pi}{2} \end{cases} \Rightarrow \frac{1}{2}t = \begin{cases} \frac{\pi}{2} + 1.41 \\ -\frac{\pi}{2} + 1.41 \end{cases}$$

$$t = \begin{cases} 2\left(\frac{\pi}{2} + 1.41\right) = 5.96 \text{ sec.} \checkmark \\ 2\left(-\frac{\pi}{2} + 1.41\right) = -0.322 \text{ sec.} \end{cases}$$

Ex: Given a DE: $x''+4x'+8x = 0$; $x(0) = 1$; $x'(0) = 2$

a) Find the solution in phase – amplitude form.

b) If we think of each equation as describing a linear mass-spring system, determine how often the mass crosses the equilibrium position.

c) Find the time at which the mass first crosses the equilibrium position.

d) Estimate the time for which $|x(t)| < 1/100$

a) $p(\lambda) = \lambda^2 + 4\lambda + 8 = 0$

$\underbrace{\lambda^2 + 4\lambda + 4} + 4 = 0$

$(\lambda+2)^2 = -4$

$\lambda + 2 = \pm 2i$

$\lambda = -2 \pm 2i$

$x(t) = A e^{-2t} \cdot \cos(2t - \delta)$

$x(0) = A\cos(-\delta) = A\cos\delta = 1.$

$x'(t) = A e^{-2t}\left[-2\cos(2t-\delta) - 2\sin(2t-\delta)\right]$

$x'(0) = A\left[-2\cos(\delta) + 2\sin(\delta)\right] = 2.$

$= -2(1) + 2A\sin\delta = 2.$

$2A\sin\delta = 4 \Rightarrow A\sin\delta = 2.$

$\begin{cases} A\cos\delta = 1 \\ A\sin\delta = 2 \end{cases} \Rightarrow \begin{matrix} A^2\cos^2\delta = 1 \\ A^2\sin^2\delta = 4 \end{matrix}$

$A^2 = 5 \Rightarrow A = \sqrt{5}.$

$\tan\delta = 2 \Rightarrow \delta = \tan^{-1}(2) = 1.107$

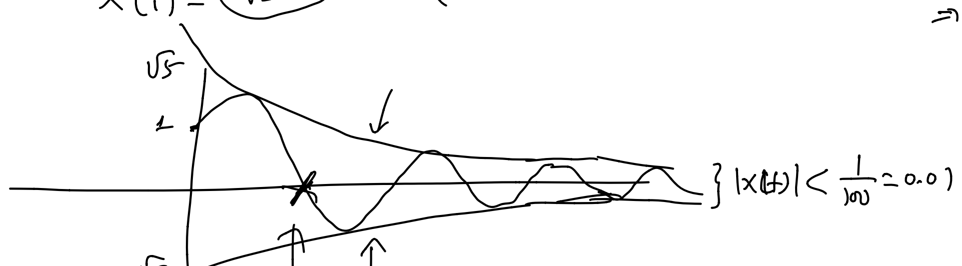
$x(t) = \sqrt{5}\, e^{-2t} \cos(2t - 1.107)$

b) period $= \frac{2\pi}{B} = \frac{2\pi}{2} = \pi. \Rightarrow$ The mass will cross the equilibrium every $\frac{\pi}{4}$ sec.

c) It crosses the 1st time $\Rightarrow$ $2t - 1.107 = \begin{cases} \frac{\pi}{2} \\ -\frac{\pi}{2} \end{cases}$

$t = \begin{cases} \frac{1}{2}\left(\frac{\pi}{2} + 1.107\right) = \boxed{1.338 \text{ sec.}} \leftarrow 1^{st}\text{ Time.} \\ \frac{1}{2}\left(-\frac{\pi}{2} + 1.107\right) = -0.23. \end{cases}$

d) $x(t) = \sqrt{5}\, e^{-2t} \cos(2t - 1.107) \Rightarrow |x(t)| = \left|\sqrt{5}\, e^{-2t}\cos(2t - 1.107)\right| < \frac{1}{100}.$



$\Rightarrow \sqrt{5}\, e^{-2t} < 0.01$

$e^{-2t} < \frac{0.01}{\sqrt{5}}$

$-2t < \ln\left(\frac{0.01}{\sqrt{5}}\right)$

$t > -\frac{1}{2}\ln\left(\frac{0.01}{\sqrt{5}}\right) \approx 2.7$ sec.

$t = 1.338$

Oscillations with External forces: (Spring/Mass Systems: Driven Motion)

Case 1: DE of Driven Motion without damping:

$$\frac{d^2y}{dt^2} + \varpi^2 y = F_0 \sin\beta t;\ \ y(0)=0;\ y'(0)=0 \quad \text{Where } F_0 \text{ is a constant and } \varpi \neq \beta$$

Sol: The homogeneous solution $y_h(t) = c_1\cos\varpi t + c_2 \sin\varpi t$ and the particular solution is

$$\left\{y_p(t) = A\cos\beta t + B\sin\beta t\right\}\left(\varpi^2\right)$$
$$\left\{y_p'(t) = -\beta A\sin\beta t + B\cos\beta t\right\}(0)$$
$$\underline{\left\{y_p'' = -\beta^2 A\cos\beta t - \beta^2 B\sin\beta\right\}(1)}$$

$$\cos\gamma t\left(\varpi^2 A - \beta^2 A\right) + \sin\beta t\left(\varpi^2 B - \beta^2 B\right) = F_0\sin\gamma t \Rightarrow \begin{cases} A\left(\varpi^2-\beta^2\right) = 0 \Rightarrow A = 0\ b/c\ \varpi\neq\beta \\ B\left(\varpi^2-\beta^2\right) = F_0 \Rightarrow B = \dfrac{F_0}{\varpi^2-\beta^2} \end{cases}$$

$$\Rightarrow y(t) = c_1\cos\varpi t + c_2\sin\varpi t + \frac{F_0}{\varpi^2-\beta^2}\sin\beta t$$

$$\Rightarrow y(0) = c_1 = 0$$

$$\Rightarrow y'(t) = \varpi c_2\cos\varpi t + \frac{\beta F_0}{\varpi^2-\beta^2}\cos\beta t \Rightarrow x'(0) = \varpi c_2 + \frac{\beta F_0}{\varpi^2-\gamma^2} = 0 \Rightarrow c_2 = -\frac{\beta F_0}{\varpi\left(\varpi^2-\beta^2\right)}$$

$$\Rightarrow y(t) = \frac{F_0}{\varpi\left(\varpi^2-\beta^2\right)}\left(\varpi\sin\beta t - \beta\sin\varpi t\right) \quad \beta\neq\varpi$$

Although the above equation is not defined for $\gamma = \varpi$, it is interesting to observe that it's limiting value as $\beta\to\varpi$ can be obtained by applying L'Hôpital's rule. This limiting process is analogous to "tuning in" the frequency of the driving force $\gamma/2\pi$ to the frequency of free vibrations $\varpi/2\pi$. Intuitively, we expect that over a length of time we should be able to substantially increase the amplitudes of vibration. For $\gamma = \varpi$ we define the solution to be.

$$y(t) = \lim_{\beta\to\varpi}\frac{F_0\left(\varpi\sin\beta t - \beta\sin\varpi t\right)}{\varpi\left(\varpi^2-\beta^2\right)} = F_0\lim_{\gamma\to\varpi}\frac{\dfrac{d}{d\beta}\left(\varpi\sin\beta t - \beta\sin\varpi t\right)}{\dfrac{d}{d\beta}\left(\varpi^3-\varpi\beta^2\right)} =$$

$$= F_0\lim_{\beta\to\varpi}\frac{\varpi t\cos\beta t - \sin\varpi t}{-2\varpi\beta} = F_0\frac{\varpi t\cos\varpi t - \sin\varpi t}{-2\varpi^2}$$

$$= \frac{F_0}{2\varpi^2}\sin\varpi t - \frac{F_0}{2\varpi}t\cos\varpi t$$

Clearly, $\lim\limits_{t\to\infty}\left|x(t)\right| = \lim\limits_{t\leftarrow\infty}\left|\left(\frac{F_0}{2\varpi^2}\sin\varpi t - \frac{F_0}{2\varpi}t\cos\varpi t\right)\right| = \infty$ for $t_n = \frac{n\pi}{\varpi}$, $n = 1,2,3,\ldots$

Pure Resonance: Although equation $x(t)=\frac{F_0}{\varpi\left(\varpi^2-\beta^2\right)}(\varpi\sin\beta t-\beta\sin\varpi t)$ is not defined for $\beta=\varpi$

Ex: $y''+25x=10\cos 5t;\ y(0)=0;\ y'(0)=1$

Sol: Clearly, for homogeneous part, we have $y_h(t)=C_1\cos 5t+C_2\sin 5t$ and particular

$$\left\{y_p(t)=t\left(A\cos(5t)+B\sin(5t)\right)\right\}(25)$$

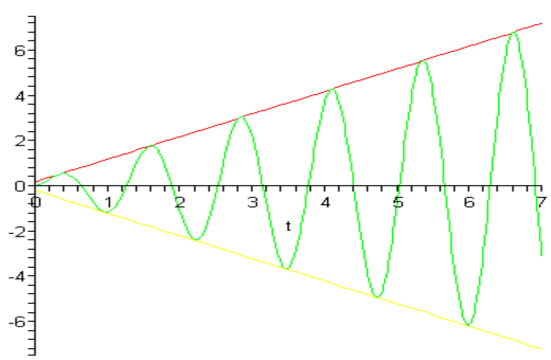
$$\left\{y_p'(t)=A\cos(5t)+B\sin(5t)+t\left(-5A\sin(5t)+5B\cos(5t)\right)\right\}(0)$$

$$\left\{\begin{array}{c} y_p''(t)=-5A\sin(5t)+5B\cos(5t)-5A\sin(5t)+5B\cos(5t) \\ +t\left(-25A\cos(5t)-25B\sin(5t)\right)\end{array}\right\}(1)$$

$$\text{solution }=\left\{\begin{array}{l} t\left[(25A-25A)\cos(5t)+(25B-25B)\sin(5t)\right] \\ +(5B+5B)\cos(5t)+(-5A-5A)\sin(5t)\end{array}\right\}=10\cos(5t)$$

$$\Rightarrow 10B=10\Rightarrow B=1;\ -10A=0\Rightarrow A=0$$

$$y(t)=y_h+y_p=C_1\cos 5t+C_2\sin 5t+t\sin(5t)\left\{\begin{array}{l} y(0)=0\Rightarrow c_1=0 \\ y'(0)=1\Rightarrow c_2=\frac{1}{5}\end{array}\right.\Rightarrow y(t)=\left(t+\frac{1}{5}\right)\sin(5t)$$



Ex: Let's now take a look at $x''+25x = 11\cos 6t;\ x(0) = 0;\ x'(0) = 0$

- Although the initial conditions are both zero, the solution is not the zero function. In contrast, if there is no forcing, then zero initial conditions imply a zero solution.

- Although the oscillations are more complicated than those of the simple sine or cosine function, the solution is periodic with a period that is slightly larger than 6.

Sol: $x_h(t) = C_1 \sin(5t) + C_2 \cos(5t);\ x_p(t) = A\cos(6t) + B\sin(6t);$

$$x'_p(t) = -6A\sin 6t + 6B\cos 6t$$

$$x_p''(t) = -36A\cos 6t - 36B\sin 6t$$

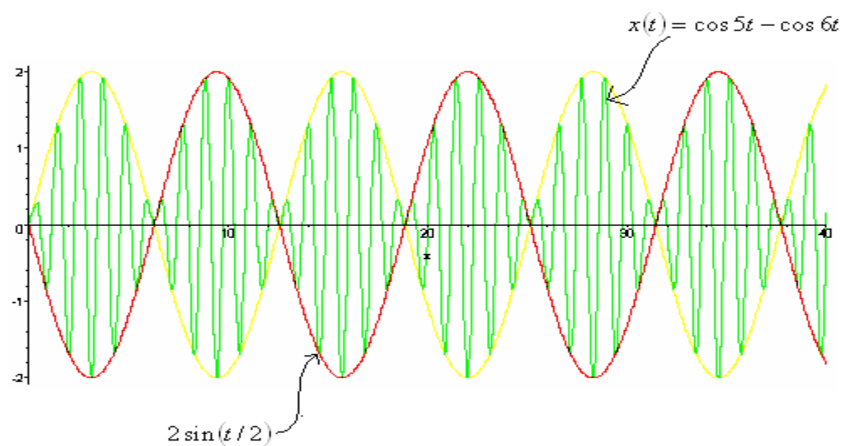
$$-36A\cos 6t - 36B\sin 6t + 25(A\cos 6t + B\sin 6t) = 11\cos 6t$$

$$\Rightarrow \begin{cases} -36A + 25A = 11 \Rightarrow -11A = 11 \Rightarrow A = -1 \\ -36B + 25B = 0 \Rightarrow B = 0 \end{cases} \Rightarrow x(t) = c_1 \sin 5t + c_2 \cos 5t - \cos 6t$$

With the initial conditions we have

$$x(0) = c_2 - 1 = 0 \Rightarrow c_2 = 1;\ x'(t) = -5c_1 \cos 5t - 5c_2 \sin 5t - 6\sin 6t$$

$$x'(0) = -5c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow x(t) = \cos 5t - \cos 6t$$



Ex: Interpret and solve the initial-value problem:

$$\frac{1}{5}\frac{d^2y}{dt^2}+1.2\frac{dy}{dt}+2y=5\cos(4t);\ y(0)=\frac{1}{2},\ y'(0)=0$$

Quiz # 4 (Individual).

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1) Solve DE n^{th} order. { - undetermined Coeff. - Annihilator. - Variation of Parameter. }

2) a) Cauchy - DE. $y = x^r$.

b) Cauchy with variation of parameter.

3) System of DE. { elimination & subs. }

4) Oscillation { 1) Harmonic. 2) Over damped. 3) Critically damped. 4) Under damped,

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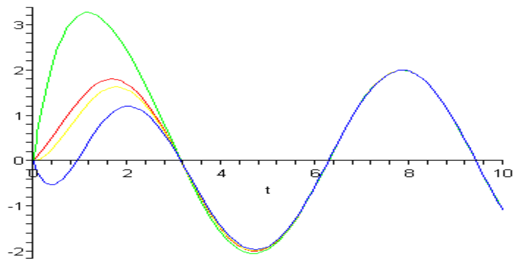
Extended → Thursday.

Ex: Solve: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 4\cos t + 2\sin t;\ x(0) = 0;\ x'(0) = m$

Where m is constant, the solution will be: $x(t) = (m-2)e^{-t}\sin t + 2\sin t$



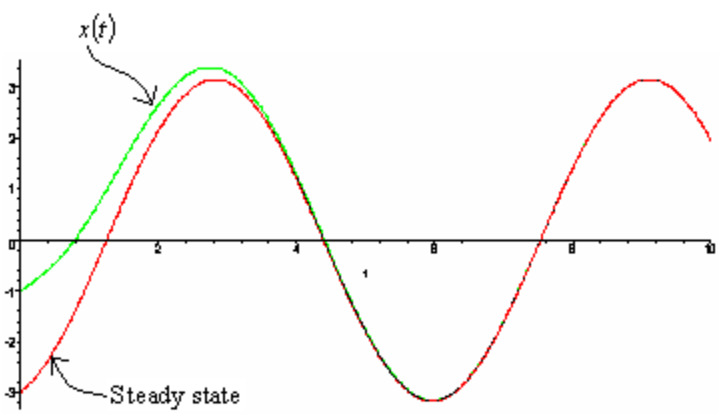
Ex: Let $x''+3x'+2x = 10\sin t;\ x(0) = -1;\ x'(0) = 1$

Sol: $p(\lambda) = \lambda^2 + 3r + 2 = 0 \Rightarrow \lambda = -1, -2; \Rightarrow x_h(t) = C_1e^{-t} + C_2e^{-2t}$

With initial condition we have the solution: $x(t) = 4e^{-t} - 2e^{-2t} + \sin t - 3\cos t$.

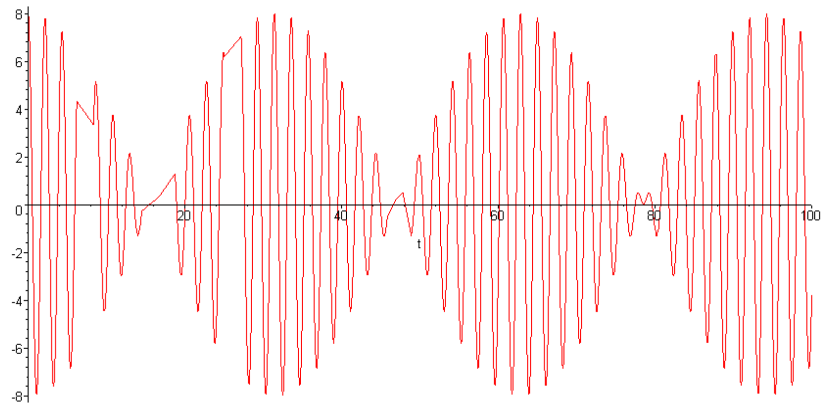
Note:

- The solution for the above solution may be regarded as the sum of a transient and a steady-state function. This is a basic characterization of the solutions of damped linear oscillators with periodic forcing. The transient function consists of those terms that tend to 0 as $t \to \infty$. In this case, the transient function is $x_T(t) = 4e^{-t} - 2e^{-2t}$ and the steady-state function is $x_{ss}(t) = \sin t - 3\cos t$. In general, the transient part of the solution is the solution of the homogeneous solution that is of course the solution for the auxiliary equation.
- The frequency of th steady-state solution is the same as the frequency of the forcing. In other words, the forcing function and the steady state are periodic with period 2π



What if different parts of Steady – State solution has different period.

Let look at the function $f(t) = 4\cos 2.8t + 4\cos 3t$. The graph of this function as below



Now, let's investigate how to find the period of this function, clearly, the period of this function contains two periods such as $T_1 = 2\pi / 2.8 = 5\pi / 7;\ and\ T_2 = 2\pi / 3$. Note that the period of a function satisfies $f(t_0 + s) = f(t_0)$. If s is a positive integer multiple of both $T_1\ and\ T_2$ the period of f is the least common multiple of $T_1\ and\ T_2$. Therefore, we seek the smallest positive integers m and n such that

$$mT_1 = nT_2 \Leftrightarrow m\left(\frac{5\pi}{7}\right) = n\left(\frac{2\pi}{3}\right) \Leftrightarrow \frac{5m}{7} = \frac{2n}{3} \Leftrightarrow \frac{m}{n} = \frac{14}{15} \Rightarrow m = 14, n = 15 \Rightarrow \text{the period}$$

$$p = 14T_1 = 15T_2 = 10\pi$$