Section 7.3 The First Shifting Theorem

- Evaluating $L(e^{5t}t^3)$ *and* $L(e^{-2t}\cos 4t)$ is straightforward provided that we known (and we do) $L(t^3)$ *and* $L(\cos 4t)$. In general, if we know the Laplace transform of a function f $L(f) = F(s)$, it is possible to compute the Laplace transform of an exponential multiple of f (t)- that is, possible to compute the Laplace transform of an exponential multiple of f, that is, $L(e^{at} f(t))$- with no additional effort other than *translating*, or *shifting*, the transform $F(s)$ *to* $F(s-a)$. This result is known as the **first translation theorem** or **fist shifting theorem.**

First Translation Theorem: (Translation on the s – axis)
If $L(f(t)) = F(s)$ and a is any real number, then $L(e^{at} f(t)) = F(s-a)$

If $\mathcal{L}(f(t)) = F(s)$ Then
$\mathcal{L}(e^{at}\cdot f(t)) = F(s-a)$
$\Rightarrow \mathcal{L}^{-1}(F(s-a)) = e^{at}\cdot f(t)$

$\mathcal{L}(t^5) = \frac{5!}{s^6}$
$\mathcal{L}(e^{3t}\cdot t^5) = \frac{5!}{(s-3)^6}$

$\mathcal{L}(\cos(3t)) = \frac{s}{s^2+9}$
$\mathcal{L}(e^{-7t}\cos(3t)) = \frac{s+7}{(s+7)^2+9}$

Proof: $L(e^{at} f(t)) = \int_0^\infty e^{-st}e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a)$

In general:

1. $L(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}; \Rightarrow L^{-1}\left(\dfrac{n!}{(s-a)^{n+1}}\right) = e^{at}t^n$

2. $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}; \Rightarrow L^{-1}\left(\dfrac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cdot\cos(bt)$

3. $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}; \Rightarrow L^{-1}\left(\dfrac{b}{(s-a)^2+b^2}\right) = e^{at}\sin(bt)$

4. $L(e^{at}\sinh(bt)) = \dfrac{b}{(s-a)^2-b^2}; \Rightarrow L^{-1}\left(\dfrac{b}{(s-a)^2-b^2}\right) = e^{at}\sinh(bt)$

5. $L(e^{at}\cosh(bt)) = \dfrac{s-a}{(s-a)^2-b^2}; \Rightarrow L^{-1}\left(\dfrac{s-a}{(s-a)^2-b^2}\right) = e^{at}\cosh(bt)$

Note: If we consider s a real variable, then the graph of $F(s-a)$ is the graph $F(s)$ is shifted a units to the right.

Ex: Evaluate the Laplace Transform of the following functions:

a) $f(t) = t^3 e^{4t}$

$$\mathcal{L}(f(t)) = \frac{3!}{(s-4)^4}$$

b) $f(t) = e^{-2t}\cos(4t)$

$$\mathcal{L}(f(t)) = \frac{s+2}{(s+2)^2+16}.$$

c) $f(t) = e^{-7t}\sin(3t) + e^{3t}(t^3 - 2t + 5) + 3$

$$f(t) = e^{-7t}\sin(3t) + e^{3t}\cdot t^3 - 2e^{3t}\cdot t + 5\cdot e^{3t} + 3.$$

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{-7t}\sin(3t)) + \mathcal{L}(e^{3t}t^3) - 2\mathcal{L}(e^{3t}\cdot t) + 5\mathcal{L}(e^{3t}) + 3\mathcal{L}(1)$$

$$= \frac{3}{(s+7)^2+9} + \frac{3!}{(s-3)^4} - 2\cdot\frac{1}{(s-3)^2} + 5\cdot\frac{1}{s-3} + 3\cdot\frac{1}{s}$$

$$e^{5t+2} = e^{5t}\cdot e^2$$

d) $f(t) = e^{5t+2}\cosh(7t) + 7e^{-3t}\sinh(2t)$

$$\mathcal{L}(f(t)) = e^2\,\mathcal{L}(e^{5t}\cosh(7t)) + 7\mathcal{L}(e^{-3t}\sinh(2t)).$$

$$= e^2\cdot\frac{s-5}{(s-5)^2-49} + 7\,\frac{2}{(s+3)^2-4}$$

Ex: Determine $L^{-1}(F(s))$ for the following:

a) $F(s)=\frac{s}{s^2+6s+11} = \frac{s+3-3}{(s+3)^2+2} = \frac{s+3}{(s+3)^2+2} - \frac{3}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(s+3)^2+2}$

Completing Square of $s^2+6s+11$

$a^2+2ab+b^2=(a\pm b)^2$

$s^2+6s+11 = \underbrace{s^2+2s\cdot 3+9}_{(s+3)^2}+2$

$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{s+3}{(s+3)^2+2}\right) - \frac{3}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{(s+3)^2+2}\right)$

$= e^{-3t}\cos(\sqrt{2}t) - \frac{3}{\sqrt{2}}e^{-3t}\sin(\sqrt{2}t)$.

$(s+4)(s-2)$

b) $F(s)=\frac{1}{(s-1)^3}+\frac{1}{s^2+2s-8} = \frac{1}{2!}\frac{2!}{(s-1)^3} + \frac{1}{3}\frac{3}{(s+1)^2-9}$

$s^2+2s-8 = \underbrace{s^2+2s+1}-9$

$= (s+1)^2-9$

$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2!}{(s-1)^3}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+1)^2-9}\right)$

$= \frac{1}{2}e^{t}t^2 + \frac{1}{3}e^{-t}\sinh(3t)$

c) $F(s)=\frac{2s+5}{(s-3)^2} = \frac{2(s-3)+6+5}{(s-3)^2} = \frac{2}{(s-3)} + \frac{11}{(s-3)^2}$

$f(t) = \mathcal{L}^{-1}(F(s)) = 2\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + 11\mathcal{L}\left(\frac{1}{(s-3)^2}\right)$

$= 2e^{3t} + 11e^{3t}\cdot t$.

d) $L^{-1}\left(\overbrace{\frac{s/2+5/3}{s^2+4s+6}}^{=F(s)}\right)=$

$s^2+4s+6 = \underbrace{s^2+4s+4}+2$

$= (s+2)^2+2$

$F(s) = \frac{1}{2}\cdot\frac{s+2-2}{(s+2)^2+2} + \frac{5}{3}\cdot\frac{1}{(s+2)^2+2}.$

$= \frac{1}{2}\cdot\frac{s+2}{(s+2)^2+2} - \frac{1}{(s+2)^2+2} + \frac{5}{3}\cdot\frac{1}{(s+2)^2+2}$

$= \frac{1}{2}\cdot\frac{s+2}{(s+2)^2+2} + \frac{2}{3}\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+2)^2+2}.$

$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2}e^{-2t}\cos(\sqrt{2}t) + \frac{2}{3\sqrt{2}}e^{-2t}\sin(\sqrt{2}t)$

e) $F(s)=\frac{s^2-17s+32}{s^3-6s^2+16s}$

$F(s) = \frac{s^2-17s+32}{s(s^2-6s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2-6s+16}.$

$s^2-17s+32 = A(s^2-6s+16) + (Bs+C)s.$

$= (A+B)s^2 + (-6A+C)s + 16A.$

$16A = 32$
$A = 2$
$-6A+C = -17$
$C = -5$
$A+B = 1$
$B = -1$

$F(s) = \frac{2}{s} + \frac{-s-5}{s^2-6s+16} = \frac{2}{s} - \frac{s-3+8}{(s-3)^2+7}$

$= \frac{2}{s} - \frac{s-3}{(s-3)^2+7} + \frac{8}{\sqrt{7}}\frac{\sqrt{7}}{(s-3)^2+7}$

$f(t) = 2\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s-3}{(s-3)^2+7}\right) + \frac{8}{\sqrt{7}}\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{(s-3)^2+7}\right)$

$= 2(1) - e^{3t}\cos(\sqrt{7}t) + \frac{8}{\sqrt{7}}e^{3t}\sin(\sqrt{7}t)$

f) $F(s)=\frac{3s^2+14s+20}{s^2+4s+9}$

$F(s) = \frac{3(s^2+4s+9)+2s-7}{s^2+4s+9}$

$s^2+4s+9 = s^2+4s+4+5 = (s+2)^2+5.$

$F(s) = 3 + \frac{2(s+2)-4-7}{(s+2)^2+5}$

$= 3 + 2\cdot\frac{s+2}{(s+2)^2+5} - \frac{11}{\sqrt{5}}\cdot\frac{\sqrt{5}}{(s+2)^2+5}$

$\mathcal{L}^{-1}(F(s)) = 3\mathcal{L}^{-1}(1) + 2\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+5}\right) - \frac{11}{\sqrt{5}}\mathcal{L}^{-1}\left(\frac{\sqrt{5}}{(s+2)^2+5}\right)$

$= \frac{3}{s}$?? $2e^{-2t}\cos(\sqrt{5}t) - \frac{11}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)$

Ex: Using Laplace Transform to solve the following:

a) $y''+4y'+6y=1+e^{-t};\ y(0)=0;\ y'(0)=0$

$$\mathcal{L}(y'') + 4\mathcal{L}(y') + 6\mathcal{L}(y) = \mathcal{L}(1) + \mathcal{L}(e^{-t})$$

$$\left(s^2\mathcal{L}(y) - sy(0) - y'(0)\right) + 4\left(s\mathcal{L}(y) - y(0)\right) + 6\mathcal{L}(y) = \frac{1}{s} + \frac{1}{s+1}$$

$y(0) = 0$, $y'(0) = 0$

$$(s^2+4s+6)\mathcal{L}(y) = \frac{s+1+s}{s(s+1)} = \frac{2s+1}{s(s+1)}$$

$$\mathcal{L}(y) = \frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4s+6}$$

$$2s+1 = A(s+1)(s^2+4s+6) + Bs(s^2+4s+6) + (Cs+D)(s^2+s)$$

$$= (A+B+C)s^3 + (A+4A+4B+D+C)s^2$$

$$+ (6A+4A+6B+D)s + 6A$$

where $B\Big|_{s=-1} = \frac{-1}{-(1-4+6)} = \frac{1}{3}$

$$\begin{cases} 6A = 1 \Rightarrow A = \frac{1}{6} \\ A+B+C = 0 \\ 5A+4B+D+C = 0 \\ 10A+6B+D = 2 \end{cases}$$

$A+B+C=0 \rightarrow \frac{1}{6} + \frac{1}{3} + C = 0$

$C = -\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}$

$\frac{5}{3} + 2 + D = 2 \Rightarrow D = -\frac{5}{3}$

$$F(s) = \frac{1}{6}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s+1} + \frac{-\frac{1}{2}(s) - \frac{5}{3}}{s^2+4s+6}$$

$$= \frac{1}{6}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s+2-2}{(s+2)^2+2} - \frac{5}{3}\cdot\frac{1}{(s+2)^2+2}$$

$$= \frac{1}{6}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s+2}{(s+2)^2+2} - \frac{8}{3}\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+2)^2+2}$$

$$y(t) = \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos(\sqrt{2}t) - \frac{8}{3\sqrt{2}}e^{-2t}\sin(\sqrt{2}t)$$

b) $y''+2y'-3y = 26e^{2t}\cos t;\ y(0)=1;\ y'(0)=0$

$\mathcal{L}(y'') + 2\mathcal{L}(y') - 3\mathcal{L}(y) = 26\,\mathcal{L}(e^{2t}\cos t)$

$\left(s^2\mathcal{L}(y) - s\,y(0) - y'(0)\right) + 2\left(s\mathcal{L}(y) - y(0)\right) - 3\mathcal{L}(y) = 26\,\dfrac{s-2}{(s-2)^2+1}$

$y(0)=1,\quad y'(0)=0,\quad y(0)=1$ ✓

$(s^2+2s-3)\mathcal{L}(y) = \dfrac{26s-52}{s^2-4s+5} + s + 2$

$(s+3)(s-1)\mathcal{L}(y) = \dfrac{26s-52+s^3-4s^2+5s+2s^2-8s+10}{s^2-4s+5}$

$$\mathcal{L}(y) = F(s) = \boxed{\dfrac{s^3-2s^2+23s-42}{(s+3)(s-1)(s^2-4s+5)}}$$

$$= \frac{A}{s+3} + \frac{B}{s-1} + \frac{Cs+D}{s^2-4s+5}$$

$s^3-2s^2+23s-42 = A(s-1)(s^2-4s+5) + B(s+3)(s^2-4s+5) + (Cs+D)(s^2+2s-3)$

$= s^3(A+B+C) + s^2(-4A-A+3B-4B+2C+D) + s(5A+4A+5B-12B-3C+2D) - 5A+15B-3D$

$$\begin{cases} A+B+C = 1 \ ✓ \\ -5A-B+2C+D = -2 \ ✓ \\ 9A-7B-3C+2D = 23 \\ -5A+15B-3D = -42 \end{cases}$$

$A\Big|_{s=-3} = \dfrac{-27-18-69-42}{-4(9+12+5)} = \dfrac{3}{2}$

$B\Big|_{s=1} = \dfrac{1-2+23-42}{4(1-4+5)} = -\dfrac{5}{2}$

$C = 1 - \dfrac{3}{2} + \dfrac{5}{2} = 2$

$$-\frac{15}{2}+\frac{5}{2}+4+D=-2$$

$$D=-6+5=-1$$

$$F(s)=\frac{3}{2}\cdot\frac{1}{s+3}-\frac{5}{2}\cdot\frac{1}{s-1}+\frac{2s-1}{s^2-4s+5}.$$

$$=\frac{3}{2}\cdot\frac{1}{s+3}-\frac{5}{2}\cdot\frac{1}{s-1}+\frac{2(s-2)+3}{(s-2)^2+1}$$

$$=\frac{3}{2}\cdot\frac{1}{s+3}-\frac{5}{2}\cdot\frac{1}{s-1}+2\,\frac{s-2}{(s-2)^2+1}+3\cdot\frac{1}{(s-2)^2+1}$$

$$y=\mathcal{L}^{-1}(F(s))=\frac{3}{2}e^{-3t}-\frac{5}{2}e^{t}+2e^{2t}\cos t+3e^{2t}\sin t$$

1^{st} shift

Given: $\mathcal{L}(f(t))=F(s)$

$$\mathcal{L}(e^{at}\cdot f(t))=F(s-a)$$

$$\mathcal{L}^{-1}(F(s-a))=e^{at}\cdot f(t).$$

$$\mathcal{L}\left(e^{-3t}\cos(5t)\right) = \frac{s+3}{(s+3)^2+25}.$$

$$\mathcal{L}\left(\cos(5t)\right) = \frac{s}{s^2+25}$$

$$\mathcal{L}^{-1}\left(\frac{s+2}{s^2+6s+18}\right).$$

$$s^2+6s+18 = (s^2+6s+9)+9.$$

$$= (s+3)^2+3^2.$$

$$\mathcal{L}^{-1}\left(\frac{s+3-1}{(s+3)^2+3^2}\right),$$

$$= \mathcal{L}^{-1}\left(\frac{s+3}{(s+3)^2+3^2}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+3)^2+3^2}\right)$$

$$= e^{-3t}\cos(3t) - \frac{1}{3}e^{-3t}\sin(3t)$$