Email your project to ftran@mtsac.edu with your full name and class on the subject line of the email. Do not turn in a hardcopy of your project.

**Step 1:** Initialize the program:

```latex
with(DEtools):
```

**Direction Fields:**

To sketch the slope fields, say the differential equation: \( \frac{dy}{dt} = 2 \sin(xy) \)

```latex
dfieldplot(diff(y(x), x) = 2 \cdot \sin(x \cdot y(x)), y(x), x=-2 ..2, y=-5 ..5);
```

Problem 1: Use the above syntax to sketch the slope fields of the following differential equations:

a) \( \frac{dy}{dx} = 2x - 3y; \ y(0) = 2 \)  
b) \( \frac{dy}{dt} = \cos(xy) + x; \ y(1) = -1 \)

c) \( \frac{dy}{dt} = x^2 - y + 1; \ y(3) = -\frac{1}{2} \)  
d) \( \frac{dy}{dt} = \ln(x + y); \ y(1) = 0 \)

e) \( \frac{dy}{dt} = \sin(x) + \cos(y); \ y(\pi) = 2 \)  
f) \( \frac{dy}{dt} = x^2 \tan(y); \ y(0) = 1 \)
Solving Differential Equations:

Define an ODE: (There are several ways of defining equations, we start with the basic one first, just basically give the ODE a name)

Ex1: Solve the First Order Differential Equation:

\[
\frac{dy}{dt} - \frac{1}{3}y = \sin(t)
\]

> with(DEtools):
> eq1 := diff(y(t),t) - (1/3)*y(t) = sin(t);
> dsolve(eq1,y(t));

We can solve with initial values:
Set up the initial value:
> IC := y(0) = 1;
> sol := dsolve({eq1,IC},y(t));

Problem 1: Follow the outline above to solve problems 15 – 20 on page 527.

Higher Order ODE:

Example:

\[
m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = 0
\]

> eq2 := m*diff(y(t),t$2) + b*diff(y(t),t) + k*y(t) = 0;
> dsolve(eq2,y(t));

Example:

\[
\frac{d^4y}{dt^4} - 10 \frac{d^3y}{dt^3} + 35 \frac{d^2y}{dt^2} - 50y + 24 = 5e^t
\]

> eq3 := diff(y(t),t$4) - 10*diff(y(t),t$3) + 35*diff(y(t),t$2) - 50*y(t) + 24 = 5*exp(t);
> dsolve(eq3,y(t));
Solving IVP

Now, with equation #1, we want to solve it with initial value say \( y(0) = \pi \)

\[
y(t) = -\frac{9}{10} \cos(t) - \frac{3}{10} \sin(t) + e^{\left(\frac{1}{3} t\right)} \left( \pi + \frac{9}{10} \right)
\]

For equation #2, we want to solve with initial value \( y(0) = 10, \ y'(0) = 2 \)

We can define the initial value as:

\[
IV := y(0) = 10, \ D(y)(0) = 2
\]

and for equation #2, for instance, \( m =1, b = -2, k =0.1, \) we then substitute into equation #2 with new equation name:

\[
eq 2m := \text{subs}(m=1,b=-2,k=0.1,\text{eq2});
\]

\[
eq 2m := \frac{d^2}{dt^2} y(t) - 2 \left( \frac{d}{dt} y(t) \right) + 0.1 \ y(t) = 0
\]

Now, we can solve \( eq2m \) with the initial value \( IV \) as follows:

\[
y(t) = \left( 5 - \frac{4}{3} \sqrt{310} \right) e^{\left(\frac{1}{10} \left( 10 + 3 \sqrt{10} \right) t\right)} + \left( \frac{4}{3} \sqrt{10} + 5 \right) e^{\left(\frac{1}{10} \left( -10 + 3 \sqrt{10} \right) t\right)}
\]

Problem #2

Now, use the outline above to solve the following ODE:

\[ a) \quad \left(1 + e^t\right) \frac{dy}{dt} + e^t y = 0; \quad y(0) = 1 \]

\[ b) \quad \frac{dy}{dx} + y = \sin x; \quad y(1) = -\frac{\pi}{2} \]

\[ c) \quad \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 7y = 0; \quad y(0) = 0; \quad y'(0) = -1 \]

\[ d) \quad \frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 4x = \cos 2t; \quad x(0) = \pi \]

\[ e) \quad \frac{d^2 x}{dt^2} + 2x = \sin(2\sqrt{2}t); \quad x(0) = 1; \quad x'(0) = -1 \]

\[ f) \quad y''' - 3y'' + 3y' - y = x - 4e^x \]

\[ g) \quad \frac{d^2 x}{dt^2} + \frac{dx}{dt} + x = x \sin x; \quad x) \]

\[ h) \quad 2y''' - 3y'' - 3y' + 2y = \left(e^x + e^{-x}\right)^2 \]
Ex2: Suppose that you borrow $600,000 to buy a house at 5.5% annual interest rate for a period of 30 years.
a) Let $B(t)$ be the your loan balance at any time $t$ (in years) and $M$ be the monthly payment. Set up an IVP for $B(t)$.

\[
\text{IC := } B(0) = \text{loan}; \quad \text{Initial balance} \quad B(0) = 600000
\]

\[
eq \quad \text{eq := diff}(B(t), t) = \text{rate} \cdot B(t) - 12 \cdot M; \quad \text{Set up IVP}
\]

\[
\frac{dB(t)}{dt} = 0.055 B(t) - 12 M
\]

\[
sol := \text{dsolve}\{\text{eq}, \text{IC}\}, B(t)\} \quad \text{Solve the IVP}
\]

\[
B(t) = \frac{2400}{11} M + e^{\frac{11}{200} t} \left(600000 - \frac{2400}{11} M\right)
\]

\[
\text{funct := eval(rhs(sol), } t = 30); \quad \text{Find the balance 30 years later in term of } M
\]

\[
\frac{2400}{11} M + e^{\frac{33}{20}} \left(600000 - \frac{2400}{11} M\right)
\]

\[
\text{monthlypay := round(solve(funct = 0.0, } M); \quad \text{Find the monthly payment}
\]

\[
3404
\]

\[
\text{balanceFunct := eval(rhs(sol), } M = \text{monthlypay}); \quad \text{Find the balance at any time } t.
\]

\[
\frac{8169600}{11} - \frac{1569600}{11} e^{\frac{11}{200} t}
\]

\[
\text{balanceYearOne := round(eval(balanceFunct, } t = 1)); \quad \text{Find the balance after one year } B(1)
\]

\[
591932
\]

\[
\text{interestYearOne := 12 \cdot monthlypay} - (\text{loan} - \text{balanceYearOne}); \quad \text{Find the interest for year one.}
\]

\[
32780
\]

Now, suppose you want to pay off the loan faster, and you have additional $300/month.

\[
\text{prepay := monthlypay + 300;}
\]

\[
3704
\]

\[
\text{newbalance := eval(rhs(sol), } M = \text{prepay}); \quad \text{Balance function with new payment}
\]

\[
\frac{8889600}{11} - \frac{2289600}{11} e^{\frac{11}{200} t}
\]

\[
\text{newterm := round(solve(newbalance = 0, } t); \quad \text{Find when the loan is paid off with new payment.}
\]

\[
25
\]

\[
\text{newTotalInterest := 12 \cdot prepay \cdot newterm} - \text{loan};
\]

\[
511200
\]

\[
\text{AmountSave := totalInterest} - \text{newTotalInterest};
\]

\[
114240
\]
**Problem #3**: Follow the outline as above to solve the following:

- **Suppose you borrow a $500,000 to buy a house at an interest rate of 6.25% for 30 years.**
  - a) Let \( B(t) \) be the balance of your loan at time \( t \) (in years), and \( M \) be your monthly payment. Set up an IVP for \( B(t) \) and then solve for \( B(t) \) in term of \( M \).
  - b) Determine your monthly payment.
  - c) Determine the interest amount after the first year.
  - d) How much interest do you pay when you pay off the loan after 30 years?
  - e) Suppose you want to pay off the loan sooner, you put in an extra $200 per month. Determine the time to pay off the loan with this new payment.
  - f) Determine the interest you pay with this new payment and finally compare this interest with interest in (c) to determine how much you save and how much time you save.
  - g) Suppose you go with the loan with the new payment for 3 years, then a certain bank offers you an interest rate of 6% for 15 years with paper work of $5,000 (closing cost). Should you consider refinance your loan why or why not?

**Application: Mixing Problems**

**Ex3**: A tank initially contains 40 gal of sugar water having a concentration of 3 lb of sugar for each gallon of water. At time zero, sugar water with a concentration of 4 lb of sugar per gal begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume of the sugar-water solution in the tank remains constant.

(a) How much sugar is in the tank after 15 minutes?

(b) How long will it take the sugar content in the tank to reach 150 lb? 170 lb?

(c) What will be the eventual sugar content in the tank?

**Initialize**:

```maple
> restart;
> with( DEtools );
> with( plots );
```

The mathematical formulation of this problem must express the physical requirement that

\[
\frac{d}{dt} (\text{amount of sugar in tank}) = (\text{rate sugar is added to tank}) - (\text{rate sugar is removed from tank})
\]

Let \( x(t) \) denote the amount of sugar (pounds) in the tank at time \( t \) (minutes). Then, the rates in and out are respectively.

```maple
> rate_in := 4 * 2;
> rate_out := (x(t)/40) * 2;
```

so that the governing ODE is

```maple
> ode := diff( x(t), t ) = rate_in - rate_out;
```

The amount of sugar in the tank initially, that is, when \( t = 0 \), gives the initial condition

```maple
> ic := x(0)=40 * 3;
```

The solution to the IVP is

```maple
> sol := dsolve( { ode, ic }, x(t), [linear] );
```

(a) The amount of sugar (in pounds) in the tank after 15 minutes is

```maple
> eval( sol, t=15. );
```

(b) The tank will contain 150 pounds of sugar at a time \( t \) (in minutes) satisfying

```maple
> eq150 := eval( sol, x(t)=150 );
```

Thus, the desired time is found by the calculations

```maple
> t150 := solve( eq150, t ):
> t[`150 lbs`] = t150;
> `` = evalf(t150);
```

Repeating the same steps for the time when 170 pounds of sugar are in the tank leads to the equation
This complex-valued solution is clearly not physically realistic. A quick inspection of the solution, graph with

\begin{verbatim}
> plot( rhs(sol), t=0..120, title="One tank mixing problem" );
\end{verbatim}

shows that the amount of sugar in the tank reaches a steady-state limit that is well below 170 pounds. Therefore,
at no time is there ever 170 pounds of sugar in the tank.

\textit{(c)}

In (b) it was noted that the amount of sugar in the tank levels off below 170 pounds. The exact limit can be
determined from the solution by looking at the limit as $t \to \infty$, that is, at

\begin{verbatim}
> steady_state := map( Limit, sol, t=infinity );
\end{verbatim}

whose value is

\begin{verbatim}
> value( steady_state );
\end{verbatim}

Note that $x=160$ is an equilibrium solution for this ODE. However, be careful to avoid the common error of
concluding that the limit is 160 pounds because $x=160$ is an equilibrium solution.

\textbf{Problem 4:} Use the outline above to solve the following:

\begin{enumerate}
  \item \textbf{a)} Initially, a 100 – liter tank contains a salt solution with concentration 0.5 kg/liter. A fresher solution
  with concentration 0.1 kg/liter flows into the tank at the rate of 4 liter/min. The contents of the tank are kept
  well stirred, and the mixture flows out at the same rate it flows in.
    \begin{enumerate}
      \item Find the amount of salt in the tank as a function of time.
      \item Determine the concentration of salt in the tank at any time.
      \item Determine the steady – state amount of salt in the tank.
      \item Find the steady – state concentration of salt in the tank.
    \end{enumerate}
  \item \textbf{b)} At the start, 5 lbs of salt are dissolved in 20 gal of water. Salt solution with concentration 2 lb/gal is
  added at a rate of 3 gal/min, and the well – stirred mixture is drained out at the same rate of flow. How long
  should this process continue to raise the amount of salt in the tank to 25 lbs?
\end{enumerate}