Quiz #5 **Math 181** **Name:** KEY.

11/4/20

1. ***Scan your work into one single pdf file.***
2. ***Check your file includes every page that you want to turn in.***
3. ***pdf file must be readable (without any shadow or dark spots)***
4. ***This quiz has to be submitted by 7:35pm tomorrow (Thursday 11/5/20)***

Show all your work clearly. No Work, No Credit.

1. Solve the following differential equations: (10 pts)

a) $\frac{dy}{dx} = \frac{e^{2x-5y+1}}{e^{y-x-3}}$

$$\frac{dy}{dx} = e^{2x-5y+1-y+x+3}$$

$$\frac{dy}{dx} = e^{3x-6y+4} = e^{3x+4} \cdot e^{-6y}$$

$$\int \frac{dy}{e^{-6y}} = \int e^{3x+4}dx$$

$$\frac{1}{6}e^{6y} = \frac{1}{3}e^{3x+4} + C$$

$$e^{6y} = 2e^{3x+4} + C.$$

$$y = \frac{1}{6}\ln\left(2e^{3x+4} + C\right)$$

b) $\frac{dy}{dx} = x^3y^2 - 5xy^2 - 3y^2 \quad y(1) = -2$

$$\frac{dy}{dx} = y^2(x^3 - 5x - 3)$$

$$\int \frac{1}{y^2}dy = \int (x^3 - 5x - 3)dx$$

$$-\frac{1}{y} = \frac{1}{4}x^4 - \frac{5}{2}x^2 - 3x + C.$$

$$-\frac{1}{-2} = \frac{1}{4} - \frac{5}{2} - 3 + C.$$

$$C = \frac{1}{2} - \frac{1}{4} + \frac{5}{2} + 3 = \frac{2 - 1 + 10 + 12}{4} = \frac{23}{4}$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{4}x^4 - \frac{5}{2}x^2 - 3x + \frac{23}{4}$$

$$\Rightarrow y = \frac{-1}{\frac{1}{4}x^4 - \frac{5}{2}x^2 - 3x + \frac{23}{4}} = \frac{-4}{x^4 - 10x^2 - 12x + 23}.$$

2. Sketch with direction by eliminating the parameter. (8 pts)

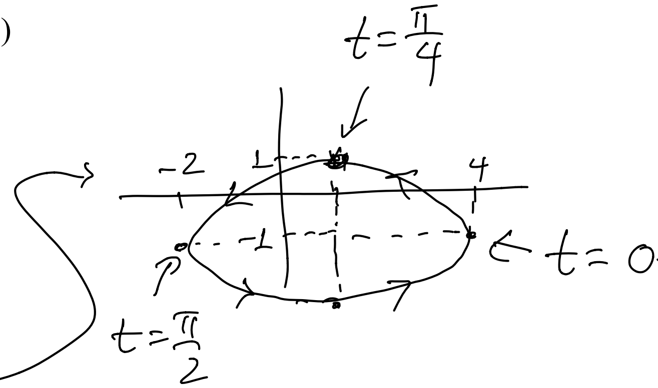
a) $\begin{cases} x = 3\cos(2t)+1 \\ y = 2\sin(2t)-1 \end{cases}; t \in R$

$\cos(2t) = \frac{x-1}{3} \Rightarrow \cos^2(2t) = \frac{(x-1)^2}{9}$

$\sin(2t) = \frac{y+1}{2} \Rightarrow \sin^2(2t) = \frac{(y+1)^2}{4}$

$+$

$1 = \frac{(x-1)^2}{9} + \frac{(y+1)^2}{4}$



$t=0 \Rightarrow \begin{cases} x=4 \\ y=-1 \end{cases}$; $t=\frac{\pi}{4} \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases}$

$t=\frac{\pi}{2} \begin{cases} x=-2 \\ y=-1 \end{cases}$

b) $\begin{cases} x = 3\sin^2 t - 1 \\ y = 2\cos t + 1 \end{cases}; t \in R$

$\sin^2 t = \frac{x+1}{3}$

$\cos^2 t = \left(\frac{y-1}{2}\right)^2$

$1 = \frac{x+1}{3} + \frac{(y-1)^2}{4}$.

$3 = x+1+\frac{3}{4}(y-1)^2$

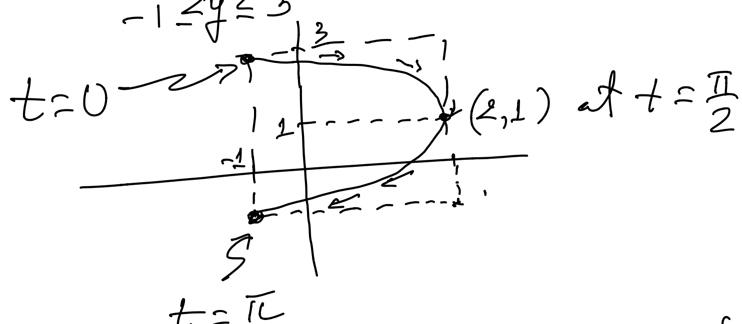
$\underbrace{x = -\frac{3}{4}(y-1)^2 + 2}_{\text{parabola with vertex: } (2,1)}$.

$x = 3\sin^2 t - 1$

$-1 \le x \le 2$.

$y = 2\cos t + 1$

$-1 \le y \le 3$



$t=0 \begin{cases} x=-1 \\ y=3. \end{cases}$; $t=\frac{\pi}{2} \Rightarrow \begin{cases} x=2 \\ y=1 \end{cases}$; $t=\pi \Rightarrow \begin{cases} x=-1 \\ y=-1 \end{cases}$

3. Find equation of tangent line to the curve $\begin{cases} x = -\sqrt{t+3} \\ y = \sqrt{3t} \end{cases}$ at $t = 3$.

Also find the value of $\dfrac{d^2y}{dx^2}$ at this point. (5pts)

Sol: Equation of tangent line $\Rightarrow$ need $\begin{cases} \text{point: } \begin{cases} x = -\sqrt{3+3} = -\sqrt{6} \\ y = \sqrt{9} = 3. \end{cases} \quad (-\sqrt{6}, 3) \\ \text{slope } m = \dfrac{dy/dt}{dx/dt}. \end{cases}$

$$m = \frac{dy/dt}{dx/dt} = \frac{\sqrt{3}\cdot\frac{1}{2}t^{-1/2}}{-\frac{1}{2}(t+3)^{-1/2}} = -\sqrt{3}\sqrt{\frac{t+3}{t}} = -\sqrt{3}\sqrt{1+\frac{3}{t}}\Big|_{t=3} = -\sqrt{3}\sqrt{4} = -2\sqrt{3}.$$

$$Y - Y_1 = m(x - x_1)$$

$$Y - 3 = -2\sqrt{3}(x + \sqrt{6})$$

Ans: $y = -2\sqrt{3}x - 2\sqrt{18} + 3.$

$$\boxed{y = -2\sqrt{3}x - 6\sqrt{2} + 3.}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[-\sqrt{3}\left(1+\frac{3}{t}\right)^{1/2}\right]}{\sqrt{3}t^{-1/2}} = \frac{-\frac{\sqrt{3}}{2}\left(1+\frac{3}{t}\right)^{-1/2}\left(-\frac{1}{t^2}\right)}{\sqrt{3}t^{-1/2}}$$

$$= \frac{1}{2}\frac{t^{1/2}}{\left(1+\frac{3}{t}\right)^{1/2}\cdot t^2}$$

$$= \frac{1}{2\left(1+\frac{3}{t}\right)^{1/2}\cdot t^{3/2}}$$

$$= \frac{1}{2\sqrt{\left(1+\frac{3}{t}\right)\cdot t^3}}$$

$$= \frac{1}{2\sqrt{t^3+3t^2}} = \frac{1}{2t\sqrt{t+3}}$$

4. Find arc-length of $\begin{cases} x = \dfrac{(2t+1)^{3/2}}{3} \\ y = 1 + \dfrac{t^2}{2} \end{cases}$ *for* $0 \le t \le 3$ (5 pts)

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{2}(2t+1)^{1/2} \cdot 2 = \sqrt{2t+1} \\ \dfrac{dy}{dt} = t \end{cases}$$

Arc-length: $L = \int ds = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\left(\frac{dx}{dt}\right)^2 = 2t+1$$

$$\left(\frac{dy}{dt}\right)^2 = t^2.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2 + 2t + 1 = (t+1)^2.$$

$$\Rightarrow L = \int_0^3 \sqrt{(t+1)^2}\, dt = \int_0^3 (t+1)\, dt = \frac{1}{2}t^2 + t \Big|_0^3 = \frac{9}{2} + 3 = \boxed{\frac{15}{2}}$$

5. Sketch the graph of $r_1 = 5$ *and* $r_2 = 10\cos(3\theta)$ on the same polar coordinate. Then setup integrals for areas of the following: (10 pts)

a) Inside r1/ outside r2

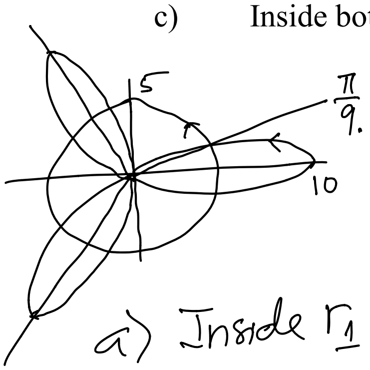
b) Inside r2/ outside r1

c) Inside both r1 and r2.

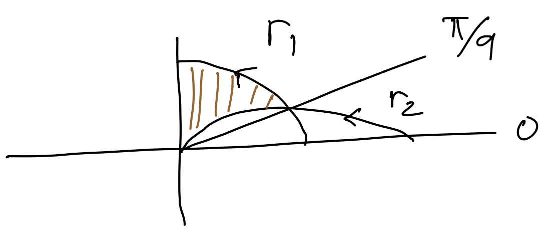
points of intersections:

$r_1 = r_2 \Rightarrow 10\cos(3\theta) = 5$

$\cos(3\theta) = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{9}$

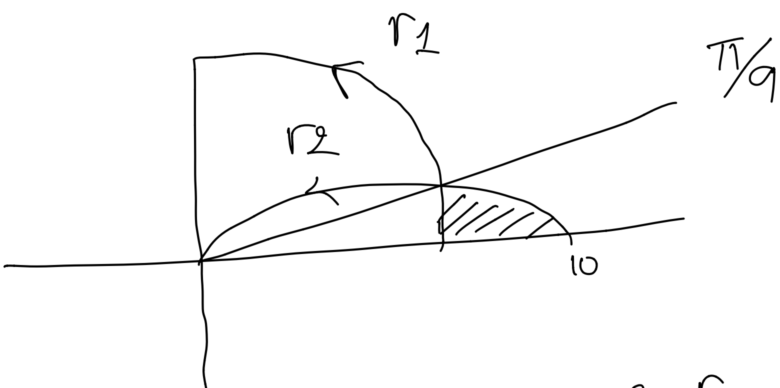


a) Inside r_1/outside r_2.



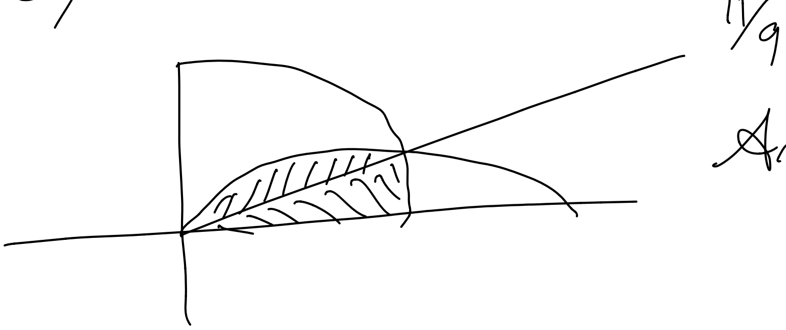
$$A = 6\left[\frac{1}{2}\int_{\pi/9}^{\pi/2} 5^2\,d\theta - \frac{1}{2}\int_{\pi/9}^{\pi/2} (10\cos(3\theta))^2\,d\theta\right]$$

b) Inside r_2/outside r_1.



$$\text{Area} = 6\left[\frac{1}{2}\int_0^{\pi/9} (10\cos(3\theta))^2\,d\theta - \frac{1}{2}\int_0^{\pi/9} 5^2\,d\theta\right]$$

c) Inside both r_1 & r_2.



$$\text{Area} = 6\left[\frac{1}{2}\int_0^{\pi/9} 5^2\,d\theta + \frac{1}{2}\int_{\pi/9}^{\pi/2} (10\cos(3\theta))^2\,d\theta\right]$$

6. Determine the limit of the following sequences: (12 pts)

a) $\{a_n\} = \left\{ \dfrac{\sqrt{3n^8 - 2n^5 + 3}}{4n^4 + n^3 + 2} \right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{3n^8 - 2n^5 + 3}}{4n^4 + n^3 + 2} \div \frac{n^4}{n^4} = \lim_{n\to\infty} \frac{\sqrt{3 - \frac{2}{n^3} + \frac{3}{n^8}}}{4 + \frac{1}{n} + \frac{2}{n^4}} = \boxed{\frac{\sqrt{3}}{4}}$$

b) $\{b_n\} = \left\{ \dfrac{(n+2)!}{(n+4)!} \right\}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{(n+2)!}{(n+4)!} = \lim_{n\to\infty} \frac{(n+2)!}{(n+4)(n+3)(n+2)!} = \lim_{n\to\infty} \frac{1}{(n+4)(n+3)} = \boxed{0}$$

c) $\{c_n\} = \left\{ \left(\dfrac{3n-2}{3n+4} \right)^{3n+2} \right\}$

Note: $\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^x = e^k$.

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \left(\frac{3n-2}{3n+4} \right)^{3n+2} = \lim_{n\to\infty} \left(\frac{1-\frac{2}{3n}}{1+\frac{4}{3n}} \right)^{3n+2}$$

$$= \left[\frac{\lim_{n\to\infty}\left(1-\frac{2}{3n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{4}{3n}\right)^n} \right]^3 \cdot \left[\lim_{n\to\infty} \left(\frac{3n-2}{3n+4} \right) \right]^2$$

$$= \left(\frac{e^{-2/3}}{e^{4/3}} \right)^3 (1)^2 = \left(\frac{1}{e^2} \right)^3 = \frac{1}{e^6}.$$

d) $\{d_n\} = \left\{ \dfrac{1}{\sqrt{2n^2+1} - \sqrt{2n^2-3n}} \right\}$

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{1}{\sqrt{2n^2+1} - \sqrt{2n^2-3n}} \cdot \frac{\sqrt{2n^2+1} + \sqrt{2n^2-3n}}{\sqrt{2n^2+1} + \sqrt{2n^2-3n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2n^2+1} + \sqrt{2n^2-3n}}{2n^2+1-(2n^2-3n)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2n^2+1} + \sqrt{2n^2-3n}}{1+3n} \div \frac{n}{n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2+\frac{1}{n^2}} + \sqrt{2-\frac{3}{n}}}{\frac{1}{n}+3} = \frac{\sqrt{2}+\sqrt{2}}{3} = \boxed{\frac{2\sqrt{2}}{3}}$$