Exam #1
Summer 2021

Thursday 7/1/21

Math 181 – Calculus II

Name: KEY

Score:_______________

1. ***You have 3 hours to finish this exam. Submitted by email will not be accepted.***
2. ***It's courtesy that you are the one who take this exam, do not seek outside help of any kind, please clear your desk, no text book, no notes, not online searching of any kind while you are taking the exam.***
3. ***Scan your exam as one pdf file and submit it thru Canvas.***
4. ***Put your Full Name clearly on the first page.***
5. ***Exam is due by 7:30pm today. Extended time 30 minutes till 8:00pm for late penalty of 25% of your score.***
6. ***Show your work clearly. No Work, No Credit.***

1. Integrate the following: (40 pts)

a) $\int \sqrt[5]{\cos^3(5x)}\sin^3(5x)\,dx$; Let $u=\cos(5x) \Rightarrow du = -5\sin(5x)dx \Rightarrow -\frac{du}{5} = \sin(5x)dx$

$$= \int \sqrt[5]{\cos^3(5x)}\left(1-\cos^2(5x)\right)\sin(5x)dx$$

$$= \int u^{3/5}(1-u^2)\cdot\frac{du}{-5} = -\frac{1}{5}\int\left(u^{3/5} - u^{13/5}\right)du$$

$$= -\frac{1}{5}\left[\frac{5}{8}u^{8/5} - \frac{5}{18}u^{18/5}\right] + C$$

$$= -\frac{1}{5}\left[\frac{5}{8}\left(\cos(5x)\right)^{8/5} - \frac{5}{18}\left(\cos(5x)\right)^{18/5}\right] + C$$

b) $\int_{1/3}^{\sqrt{3}/3} \frac{1}{(1+9x^2)\sqrt{\tan^{-1}(3x)}}dx$

Let $u = \tan^{-1}(3x)$ $\begin{cases} x = \frac{\sqrt{3}}{3} \Rightarrow u = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \\ x = \frac{1}{3} \Rightarrow u = \tan^{-1}(1) = \frac{\pi}{4} \end{cases}$

$du = \frac{3}{1+9x^2}dx \Rightarrow \frac{du}{3} = \frac{1}{1+9x^2}dx$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sqrt{u}}\cdot\frac{du}{3} = \frac{1}{3}\int_{\pi/4}^{\pi/3} u^{-1/2}du = \frac{1}{3}\,2\,u^{1/2}\Big|_{\pi/4}^{\pi/3}$$

$$= \frac{2}{3}\left[\sqrt{\frac{\pi}{3}} - \sqrt{\frac{\pi}{4}}\right] = \cdots = \#$$

c) $\int (7x^3 - 3x + 2)\sin(3x)\,dx$

$$= -\frac{1}{3}(7x^3-3x+2)\cos(3x) + \frac{1}{9}(21x^2-3)\sin(3x) + \frac{1}{27}(42x)\cos(3x) - \frac{42}{81}\sin(3x) + C$$

$7x^3-3x+2$	$(+)\ \sin(3x)\,dx$
$21x^2-3$	$(-)\ -\frac{1}{3}\cos(3x)$
$42x$	$(+)\ -\frac{1}{9}\sin(3x)$
42	$(-)\ \frac{1}{27}\cos(3x)$
0	$\frac{1}{81}\sin(3x)$

d) $\int \sin^{-1}(5x)\,dx$

$$= x\sin^{-1}(5x) - \int \frac{5x}{\sqrt{1-25x^2}}\,dx$$

let $u=\sqrt{1-25x^2}$

$u^2 = 1-25x^2$

$2u\,du = -50x\,dx$

$-\frac{1}{5}u\,du = 5x\,dx$

$\sin^{-1}(5x)$	dx
$\frac{5\,dx}{\sqrt{1-25x^2}}$	x

$$= A + \frac{1}{5}\int \frac{u\,du}{u}.$$

Ans: $= x\sin^{-1}(5x) + \frac{1}{5}\sqrt{1-25x^2} + C.$

e) $\displaystyle\int_0^8 \frac{1}{\left(1+\sqrt[3]{x}\right)^2}\,dx$

Let $u = 1+\sqrt[3]{x}$

$u-1 = \sqrt[3]{x}$

$(u-1)^3 = x \Rightarrow 3(u-1)^2\,du = dx$

$x=8 \Rightarrow u = 1+\sqrt[3]{8} = 3$

$x=0 \Rightarrow u = 1+\sqrt[3]{0} = 1$

$$= \int_1^3 \frac{1}{u^2}\cdot 3(u-1)^2\,du = 3\int_1^3 \frac{u^2-2u+1}{u^2}\,du$$

$$= 3\int_1 \left(1 - \frac{2}{u} + u^{-2}\right)du$$

$$= 3\left[u - 2\ln|u| - \frac{1}{u}\right]_1^3$$

$$= 3\left[3 - 2\ln 3 - \frac{1}{3} - 1 + 1\right] = \dots = \#$$

f) $\displaystyle\int e^{-3x}\cos(4x)\,dx$

e^{-3x}	$\cos(4x)dx$
$-3e^{-3x}dx$	$\frac{1}{4}\sin(4x)$

e^{-3x}	$\sin(4x)dx$
$-3e^{-3x}dx$	$-\frac{1}{4}\cos(4x)$

$$= \underbrace{\frac{1}{4}e^{-3x}\sin(4x)}_{A} + \frac{3}{4}\int e^{-3x}\sin(4x)\,dx$$

$$= A + \frac{3}{4}\left[-\frac{1}{4}e^{-3x}\cos(4x) - \frac{3}{4}\int e^{-3x}\cos(4x)\,dx\right]$$

$$= A - \frac{3}{16}e^{-3x}\cos(4x) - \frac{9}{16}\int e^{-3x}\cos(4x)\,dx$$

$$\left(1+\frac{9}{16}\right)\int e^{-3x}\cos(4x)\,dx = A - \frac{3}{16}e^{-3x}\cos(3x)$$

$$\Rightarrow \text{Ans}: \int e^{-3x}\cos(4x)\,dx = \frac{16}{25}\left[A - \frac{3}{16}e^{-3x}\cos(3x)\right] + C$$

g) $\int \frac{18x^2-9x+31}{(2x+1)(4x^2+9)}dx = \int \left[\frac{A}{2x+1} + \frac{Bx+C}{4x^2+9}\right]dx$

$$\Rightarrow 18x^2-9x+31 = A(4x^2+9) + (2x+1)(Bx+C)$$
$$= (4A+2B)x^2 + (B+2C)x + 9A+C$$

$4A+2B=18$

$B+2C=-9$

$9A+C=31$

$A\Big|_{x=-\frac{1}{2}} = \frac{\frac{9}{2}+\frac{9}{2}+31}{1+9} = \frac{40}{10} = 4 \Rightarrow 16+2B=18$

$B=1$

and $36+C=31 \Rightarrow C=-5$

$$\Rightarrow \int\left[\frac{4}{2x+1} + \frac{x-5}{4x^2+9}\right]dx$$
$$= \int \frac{4}{2x+1}dx + \frac{1}{8}\int \frac{8x}{4x^2+9}dx - \frac{5}{4}\int \frac{1}{x^2+\frac{9}{4}}dx.$$
$$= 2\ln|2x+1| + \frac{1}{8}\ln(4x^2+9) - \frac{5}{4}\cdot\frac{2}{3}\tan^{-1}\left(\frac{2}{3}x\right) + C.$$

h) $\int \frac{2x^3-3x^2-22x+5}{x^2+x-6}dx = \int\left(2x-5 + \frac{-5x-25}{x^2+x-6}\right)dx = \int\left(2x-5 - \frac{5x+25}{(x+3)(x-2)}\right)dx$

$$\begin{array}{r} 2x-5 \\ x^2+x-6 \,\overline{\big)\, 2x^3-3x^2-22x+5} \\ -\underline{2x^3+2x^2-12x} \\ -5x^2-10x+5 \\ -\underline{-5x^2-5x+30} \\ -5x-25 \end{array}$$

$= x^2-5x - \int\left(\frac{A}{x+3} + \frac{B}{x-2}\right)dx$

$A\Big|_{x=-3} = \frac{-15+25}{-5} = -2$; $B\Big|_{x=2} = \frac{10+25}{5} = 5$

$$= x^2-5x - \int\left(\frac{-2}{x+3} + \frac{5}{x-2}\right)dx$$

$$= x^2-5x + 2\ln|x+3| - 5\ln|x-2| + C$$

i) $\int \frac{5x+7}{\left(24-25x^2+10x\right)^{3/2}}dx$

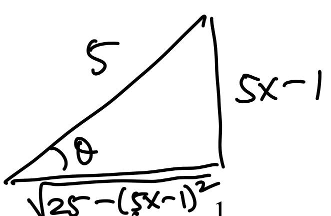
$$24-25x^2+10x = 24-\left[(5x)^2-2(5x)+1\right]+1$$
$$= 25-(5x-1)^2$$

$$= \int \frac{5x-1+8}{\left[25-(5x-1)^2\right]^{3/2}}dx \rightarrow \text{Let } 5x-1=5\sin\theta \Rightarrow 5dx = 5\cos\theta d\theta$$
$$\Rightarrow dx = \cos\theta d\theta$$

$$= \int \frac{5\sin\theta+8}{\left[25-25\sin^2\theta\right]^{3/5}}\cdot\cos\theta d\theta = \int \frac{5\sin\theta+8}{25^{3/5}\cdot(\cos^2\theta)^{3/5}}\cdot\cos\theta d\theta$$

$$= \frac{1}{125}\int \frac{5\sin\theta+8}{\cos^3\theta}\cos\theta d\theta = \frac{1}{125}\int \frac{5\sin\theta+8}{\cos^2\theta}d\theta$$

$$= \frac{1}{125}\int\left(5\sec\theta\tan\theta + 8\sec^2\theta\right)d\theta = \frac{1}{125}\left[5\sec\theta + 8\tan\theta\right]+C$$



$$= \frac{1}{125}\left[5\cdot\frac{5}{\sqrt{25-(5x-1)^2}} + 8\cdot\frac{5x-1}{\sqrt{25-(5x-1)^2}}\right]+C$$

j) $\int \frac{1}{\sqrt{x}-4\sqrt[3]{x}}dx \leftarrow$ let $x = u^6$
$dx = 6u^5 du$

$$= \int \frac{1}{\sqrt{u^6}-4\sqrt[3]{u^6}}\cdot 6u^5 du = 6\int \frac{u^5}{u^3-4u^2}du = 6\int \frac{u^3}{u-4}du$$

$$\begin{array}{r|rrrr} 4 & 1 & 0 & 0 & 0 \\ & & 4 & 16 & 64 \\ \hline & 1 & 4 & 16 & \boxed{64} \end{array} \Bigg\} = 6\int\left(u^2+4u+16+\frac{64}{u-4}\right)du$$

$$= 6\left[\frac{1}{3}u^3 + 2u^2 + 16u + 64\ln|u-4|\right]+C.$$

$$= 6\left[\frac{1}{3}\left(\sqrt[6]{x}\right)^3 + 2\left(\sqrt[6]{x}\right)^2 + 16\sqrt[6]{x} + 64\ln\left|\sqrt[6]{x}-4\right|\right]+C$$

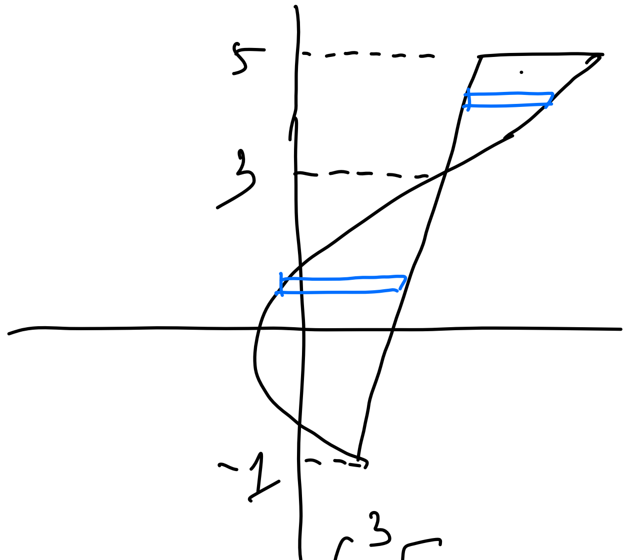
2. Sketch and then set up the integral(s) for area bounded by the following functions:
$x = 2y^2 - 3y - 2 \;\; and \;\; x - y = 4 \;\; for \;\; -1 \le y \le 5$ (5 pts)

$x = (2y+1)(y-2)$ pts of intersection: $2y^2 - 3y - 2 = y + 4$

$2y^2 - 4y - 6 = 0$

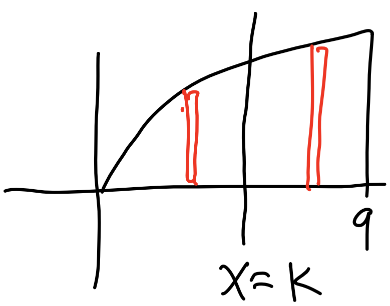
$y^2 - 2y - 3 = 0$

$(y+1)(y-3) = 0$ $\Rightarrow y = 3, -1$



$$\text{Area} = \int_{-1}^{3} \left[(y+4) - (2y^2 - 3y - 2) \right] dy + \int_{3}^{5} \left[(2y^2 - 3y - 2) - (y+4) \right] dy$$

3. Find a constant k so that the line $x = k$ bisects the area bounded by $y = \sqrt{x}, \; y = 0 \;\; and \;\; x = 9.$ (5 pts)



$$\int_0^k \sqrt{x}\,dx = \int_k^9 \sqrt{x}\,dx.$$

$k^{1/2}$

$$\frac{2}{3} x^{3/2} \Big|_0^k = \frac{2}{3} x^{3/2} \Big|_k^9$$

$$k^{3/2} = 9^{3/2} - k^{3/2} \Rightarrow 2k^{3/2} = 27$$

$$\Rightarrow k^{3/2} = \frac{27}{2} \Rightarrow k = \left(\frac{27}{2}\right)^{2/3} \approx 5.7.$$

4. Sketch and then set up the integral(s) for volumes for the region bounded by $y=-x^2-2x+3$ *and* $2x-y=2$ which is rotated about the following lines:

a) $x=1$ (5 pts)

pts of intersection: $-x^2-2x+3 = 2x-2$

$x^2+4x-5=0$

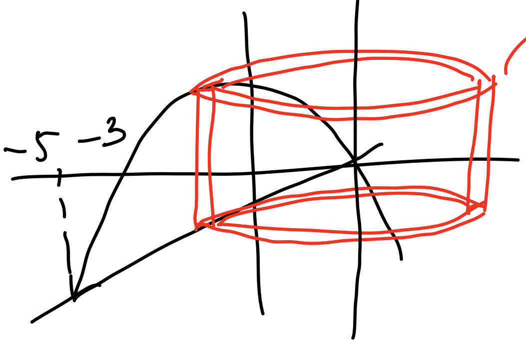
$(x+5)(x-1)=0$

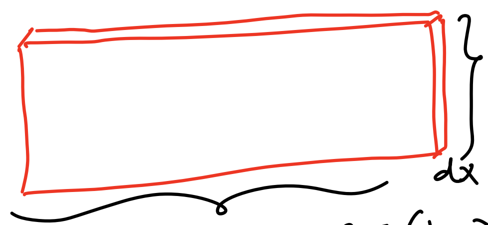
$x=1,-5$.

$y = -(x^2+2x-3)$

$= -(x+3)(x-1)$

$x=1$





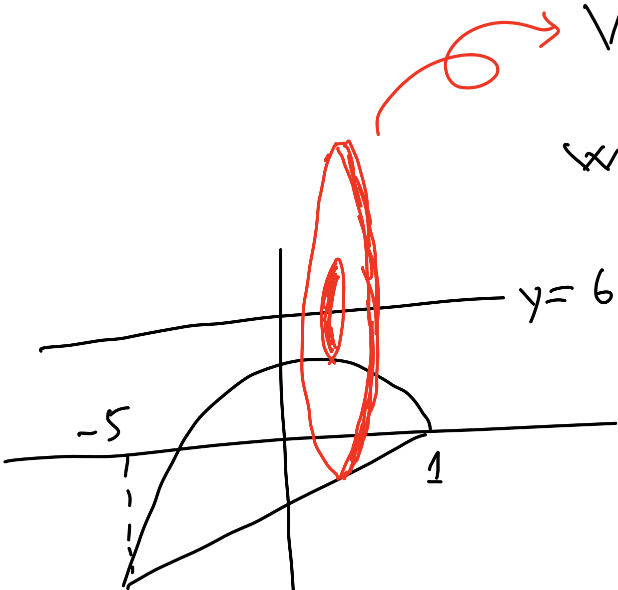
$-x^2-2x+3-(2x-2)$

$= -x^2-4x+5$

$2\pi r = 2\pi(1-x)$

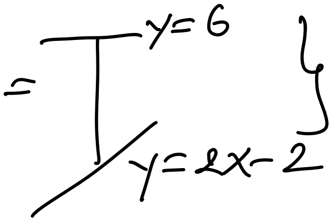
$$V = 2\pi\int_{-5}^{1}(1-x)(5-4x-x^2)\,dx$$

b) $y=6$ (5 pts)



$$V = \pi\left(R_o^2 - r_i^2\right)dx$$

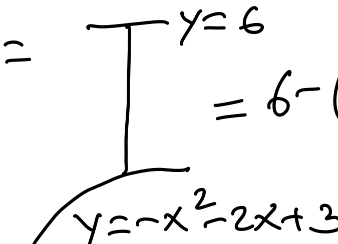
where $R_o =$



$= 6-(2x-2)$

$= 8-2x$

$r_i =$



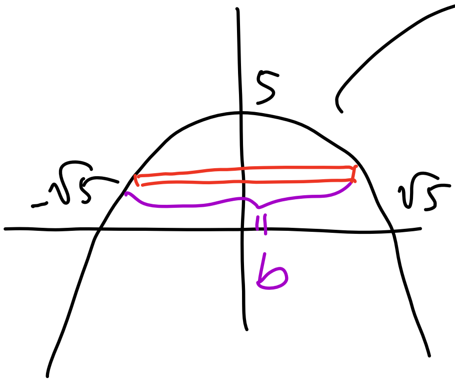
$= 6-(-x^2-2x+3)$

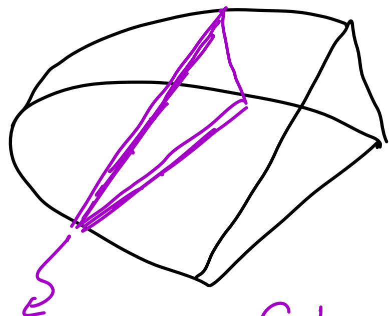
$= 3+x^2+2x$

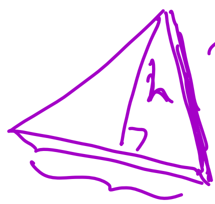
Volume: $$V = \pi\int_{-5}^{1}\left[(8-2x)^2-(3+2x+x^2)^2\right]dx$$

5. The base of a solid is bounded by $y = 5 - x^2$ *and* $y = 0$, All parallel cross sections are equilateral triangles perpendicular to the base and the y – axis. Find its volume. (5pts)

$y = 5 - x^2$







$\Rightarrow h = \frac{\sqrt3}{2} b.$

$\Rightarrow V = \frac12 b h \, dy = \frac12 b \cdot \frac{\sqrt3}{2} \cdot b \cdot dy$

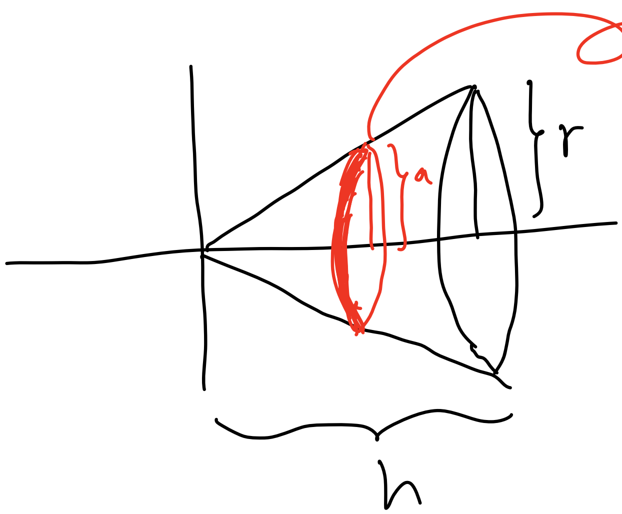
$$\begin{cases} y = 5 - x^2 \\ x^2 = 5 - y \\ x = \pm\sqrt{5-y} \end{cases}$$

$\Rightarrow V = \frac{\sqrt3}{4} b^2 dy$ where $b =$  $= x_r - x_l$

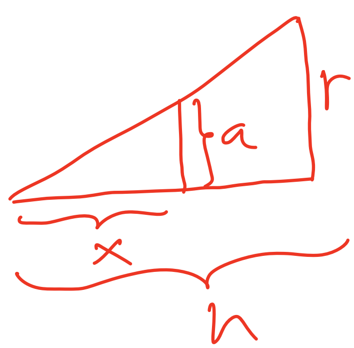
$\Rightarrow b = 2\sqrt{5-y}$

$$\Rightarrow V = \int_0^5 \frac{\sqrt3}{4}\left(2\sqrt{5-y}\right)^2 dy = \sqrt3 \int_0^5 (5-y)\,dy = \sqrt3\left(25 - \tfrac12(25)\right) = \boxed{\tfrac{\sqrt3}{2}(25)}$$

6. Prove that the volume of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$ (5pts)



$V = \pi a^2 dx$



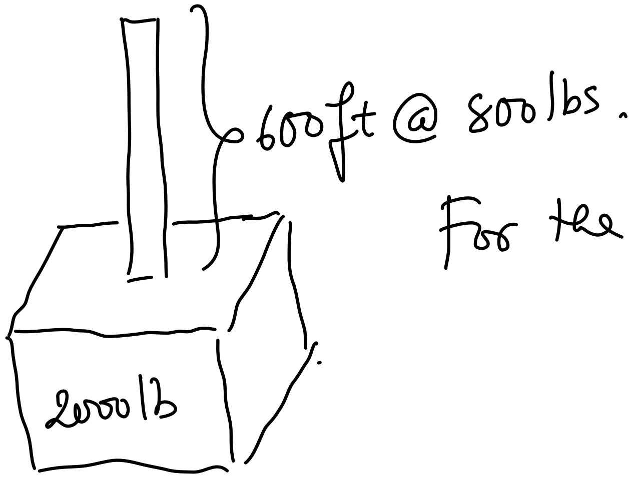
$\frac{a}{r} = \frac{x}{h}$

$\Rightarrow a = \frac{r}{h} x$

$$\Rightarrow V = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx = \frac{\pi \cdot r^2}{h^2}\int_0^h x^2 dx = \frac{\pi r^2}{h^2}\cdot \frac13 x^3 \Big|_0^h$$

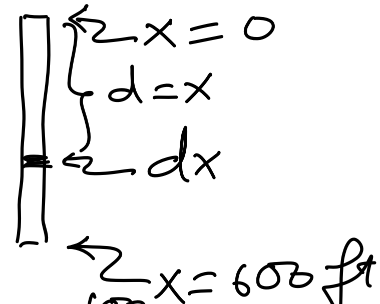
$$= \frac{\pi r^2}{h^2}\cdot\frac13\cdot h^3 = \boxed{\tfrac13 \pi r^2 h}$$

7. a) A heavy cable, 600 ft long that weights 800 lbs is attatched to a 2000 – lb – elevator. Calculate the total work that needed to lift both the cable and the elevator up 600 ft. (5pts)



For the elevator: $W_e = F \cdot d = (2000\text{ lb})(600\text{ft})$
$= 1{,}200{,}000$ ft-lb.

For the cable:



length	Force
600 ft	800 lbs.
1 ft	$\frac{800}{600} = \frac{4}{3}$ lb
⋮	
dx	$\to \frac{4}{3}dx$.

For the cable: $W_c = \int_0^{600} \frac{4}{3}x\,dx = \frac{2}{3}x^2 \Big|_0^{600} = \frac{2}{3}(600)^2$

$W_c = 240{,}000$ ft-lbs

Total Work: $W = W_e + W_c = 1{,}200{,}000 + 240{,}000$
$= 1{,}440{,}000$ ft-lbs.

b) It's required a work of 50 ft - lb to stretch a spring 10 inches beyond its natural length. Calculate the work needed to stretch the spring 15 inches beyond its natural length. (5 pts)

Find the spring-constant: $W = \int kx\,dx$

$10\text{in} = \frac{10}{12} = \frac{5}{6}$ft.

$50 = \int_0^{5/6} kx\,dx = \frac{k}{2}x^2 \Big|_0^{5/6} = \frac{k}{2}\cdot\frac{25}{36}$

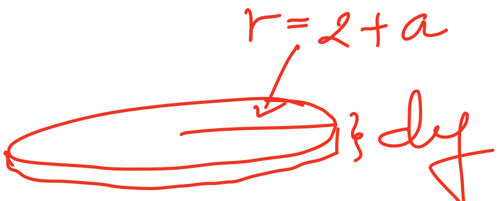
$\Rightarrow 50 = \frac{25k}{72} \Rightarrow k = 144$

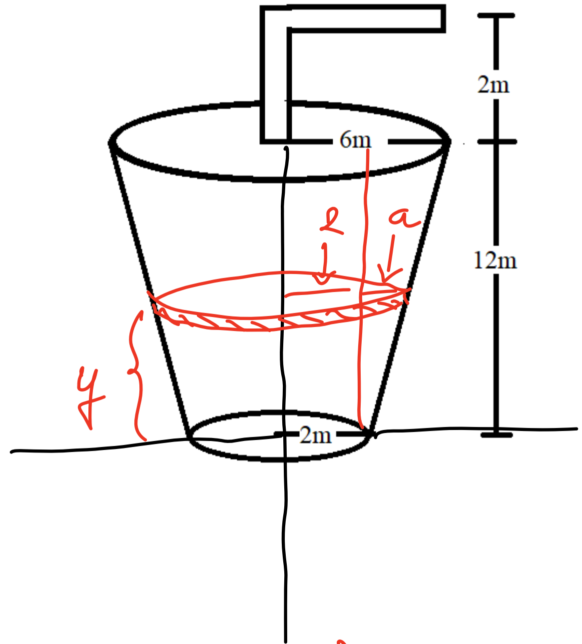
$15'' = \frac{15}{12} = \frac{5}{4}$

To stretch 15'' $\Rightarrow$ $W = \int_0^{5/4} 144x\,dx = 72x^2 \Big|_0^{5/4} = 72\left(\frac{5}{4}\right)^2$

$\Rightarrow W = \frac{72(25)}{16} = 112.5$ ft-lbs.

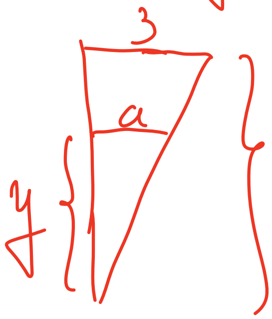
c) The following tank is full of water. Determine the work required to pump the water out of its outlet. (10 pts)





$\Rightarrow V = \pi r^2 dy = \pi(2+a)^2 dy$

where



$12 \Rightarrow \frac{a}{3} = \frac{y}{12}$

$a = \frac{1}{4}y$

$\Rightarrow V = \pi\left(2+\frac{1}{4}y\right)^2 dy = \pi\left(4+y+\frac{1}{16}y^2\right)dy$

$F = 9800\pi\left(4+y+\frac{1}{16}y^2\right)dy$

$d = 14-y$

$W = F \cdot d = 9800\pi\int_0^{12}\left(4+y+\frac{1}{16}y^2\right)(14-y)\,dy$

$W = 9800\pi\int_0^{12}\left(56-4y+14y-y^2+\frac{7}{8}y^2-\frac{1}{16}y^3\right)dy$

$= 9800\pi\left[56(12)+5(12)^2+\frac{1}{24}(12)^3-\frac{1}{64}(12)^4\right]$

$= 35{,}097{,}873$ J

8. Evaluate the following improper integrals. (10 pts)

a) $\int_0^\infty \frac{x}{(4x^2+7)^{3/5}}\,dx$ (3pts)

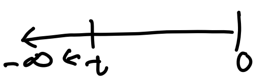
$$= \lim_{t\to\infty}\int_0^t \frac{x}{(4x^2+7)^{3/5}}\,dx \quad \left\{ \begin{array}{l} \text{let } u = 4x^2+7 \Rightarrow du = 8x\,dx \\ \Rightarrow \frac{du}{8} = x\,dx \end{array}\right.$$

$$= \lim_{t\to\infty}\int \frac{du/8}{u^{3/5}} = \frac{1}{8}\lim_{t\to\infty}\int u^{-3/5}\,du = \frac{1}{8}\lim_{t\to\infty}\frac{5}{2}u^{2/5}$$

$$= \frac{5}{16}\lim_{t\to\infty}(4x^2+7)^{2/5}\Big|_0^t = \frac{5}{16}\lim_{t\to\infty}\left[(4t^2+7)^{2/5} - 7^{2/5}\right] = \infty$$

divergent

b) $\int_{-\infty}^0 \frac{\sqrt[3]{\tan^{-1}(5x)}}{1+25x^2}\,dx$ (4pts)



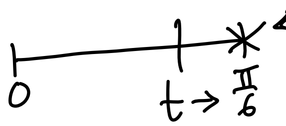
$$= \lim_{t\to-\infty}\int_t^0 \frac{\sqrt[3]{\tan^{-1}(5x)}}{1+25x^2}\,dx \quad \left\{ \begin{array}{l} \text{let } u = \tan^{-1}(5x) \\ du = \frac{5}{1+25x^2}\,dx \Rightarrow \frac{du}{5} = \frac{1}{1+25x^2}\,dx \end{array}\right.$$

$$= \lim_{t\to-\infty}\int \sqrt[3]{u}\cdot\frac{du}{5} = \frac{1}{5}\lim_{t\to-\infty}\int u^{1/3}\,du = \frac{1}{5}\lim_{t\to-\infty}\frac{3}{4}\left(\tan^{-1}(5x)\right)^{4/3}\Big|_t^0 = \frac{3}{20}\lim_{t\to-\infty}\left[0 - \left(\tan^{-1}(5t)\right)^{4/3}\right]$$

$$= \frac{3}{20}\left(0 - \left(-\frac{\pi}{2}\right)^{4/3}\right) = -\frac{3}{20}\cdot\sqrt[3]{\frac{\pi^4}{16}} \leftarrow \text{Convergent}$$

c) $\int_0^{\pi/6} \sec^2(3x)\,dx$ (4pts)

$\sec^2\left(3\cdot\frac{\pi}{6}\right) = \sec^2\left(\frac{\pi}{2}\right) = \text{undefined.}$



$$= \lim_{t\to\frac{\pi}{6}^-}\int_0^t \sec^2(3x)\,dx = \lim_{t\to\frac{\pi}{6}^-}\frac{1}{3}\tan(3x)\Big|_0^t$$

$$= \frac{1}{3}\lim_{t\to\frac{\pi}{6}^-}\left[\tan(3t) - \tan(0)\right] = \infty \quad \text{divergent}$$