Exam #2

Summer 2021

Thursday 7/15/21

Math 181 – Calculus II

Name:________________

Score:________________

Rules for taking exam:

1. Please put away all lecture notes, textbooks, or any electronic devices.
   (Except you are using iPad to take exam)
2. Only scientific calculators are allowed.
3. Scan all your work as one pdf file and submit it thru canvas.
   (Put the problem numbers in order and in the right direction (No side way or upside down). Double check if it includes all the pages that you want to submit, check your pdf file if it's readable, make sure there's no shade, shadow or black out that causes unreadable.)
4. All late submitted papers will be penalized 25% of your scores.

1. Test for convergence / divergence: (15 pts)

a) $\displaystyle\int_1^\infty \frac{\sqrt{3x^4+2x+1}}{7x^5+3x^2+2}dx \le \int_1^\infty \frac{\sqrt{3x^4+2x^4+x^4}}{x^5}dx = \int_1^\infty \frac{\sqrt{6x^4}}{x^5}dx = \sqrt{6}\int_1^\infty \frac{x^2}{x^5}dx$

Dominant terms: $\displaystyle\frac{\sqrt{x^4}}{x^5} = \frac{x^2}{x^5} = \frac{1}{x^3}$

$$= \sqrt{6}\int_1^\infty \frac{1}{x^{3>1}}dx \quad \left\{ \begin{array}{l} p=3>1 \text{ is} \\ \text{convergent by} \\ p\text{-Test} \end{array} \right.$$

$\therefore$ by C.T.T. $\Rightarrow \displaystyle\int_1^\infty f(x)dx$ is convergent.

b) $\displaystyle\int_2^\infty \frac{3x^2+7x+3}{\sqrt[3]{5x^5+x^2+1}}dx \geq \int_2^\infty \frac{x^2}{\sqrt[3]{5x^5+x^5+x^5}}dx = \int_2^\infty \frac{x^2}{\sqrt[3]{7x^5}}dx = \frac{1}{\sqrt[3]{7}}\int_2^\infty \frac{x^2}{\sqrt[3]{x^5}}dx$

Dominant term: $\displaystyle\frac{x^2}{\sqrt[3]{x^5}} = \frac{1}{x^{\frac{5}{3}-2}} = \frac{1}{x^{-1/3<1}}$

$$= \frac{1}{\sqrt[3]{7}}\int_2^\infty \frac{1}{x^{\frac{5}{3}-2}}dx = \frac{1}{\sqrt[3]{7}}\int_2^\infty \frac{1}{x^{-1/3}}dx \quad \left\{ \begin{array}{l} p=-\frac{1}{3}<1 \Rightarrow \\ \text{Divergent} \\ \text{by } p\text{-Test} \end{array} \right.$$

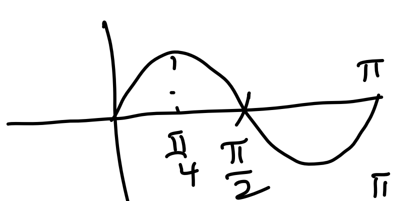
$\therefore$ by C.T.T. $\Rightarrow \displaystyle\int_2^\infty f(x)dx$ is divergent

c) $\displaystyle\int_0^{\pi/4} \frac{1}{x^3\sin(2x)dx}$

for $y = \sin(2x) \Rightarrow$

$0 \le \sin(2x) \le 1$

$0 \le x^3\sin(2x) \le x^3$



$\Rightarrow \displaystyle\frac{1}{x^3\sin(2x)} \geq \frac{1}{x^3}$

$\Rightarrow \displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{x^3\sin(2x)}dx \geq \int_0^{\pi/4} \frac{1}{x^3}dx$

$$\to \lim_{t\to 0^+}\int_t^{\pi/4} x^{-3}dx = \lim_{t\to 0^+} \frac{x^{-2}}{-2}\Bigg|_t^{\pi/4}$$

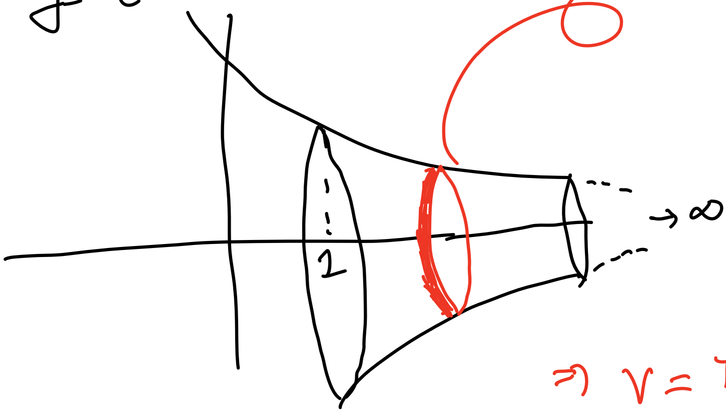
$$= -\frac{1}{2}\lim_{t\to 0^+}\left[\left(\frac{\pi}{4}\right)^{-2} - t^{-2}\right]$$

$$= -\frac{1}{2}\lim_{t\to 0^+}\left[\frac{16}{\pi^2} - \frac{1}{t^2}\right] = \infty.$$

$\therefore$ by C.T.T. $\Rightarrow \displaystyle\int_0^{\pi/4} \frac{1}{x^3\sin(2x)}dx$ is divergent.

2. The region bounded by $y = e^{-3x}$ and $y = 0$ for $x \geq 1$ is rotated about the x – axis. Calculate its volume. (5 pts)

$y = e^{-3x}$



$V = \pi r^2 dx$

where r is $= e^{-3x} - 0 = e^{-3x}$

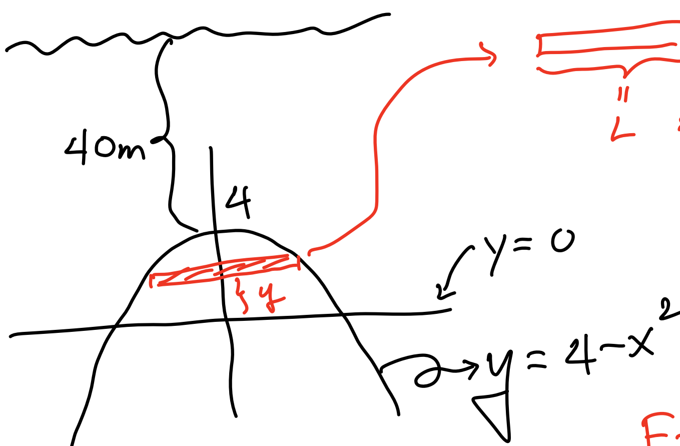
$V = \pi \int_1^\infty (e^{-3x})^2 dx$

$\Rightarrow V = \pi \lim_{t\to\infty} \int_1^t e^{-6x} dx = -\frac{\pi}{6} \lim_{t\to\infty} \left[e^{-6x}\right]_1^t.$

$\Rightarrow V = -\frac{\pi}{6} \lim_{t\to\infty} \left[e^{-6t} - e^{-6}\right] = \boxed{\frac{\pi}{6e^6}}$ convergent

$\frac{1}{e^{6t}} \to 0$

3. A vertical parabolic plate bounded by $y = 4 - x^2$ and $y = 0$ is submerged under water 40 m from its top. Calculate the hydrostatic force on the gate. (5 pts)



$L = x_r - x_L = 2\sqrt{4-y}$

$A = 2\sqrt{4-y} \cdot dy$.

$d = 44 - y$

$V = Ad = 2\sqrt{4-y}\,(44-y)\,dy$

$F = 9800 \cdot 2 \int_0^4 \sqrt{4-y}\,(44-y)\,dy$

$\Rightarrow x^2 = 4 - y$.

$x = \pm\sqrt{4-y}$

let $u = \sqrt{4-y}$ $\begin{cases} y=4 \Rightarrow u=0 \\ y=0 \Rightarrow u=2 \end{cases}$ $\Rightarrow u^2 = 4-y \Rightarrow y = 4-u^2$

$dy = -2u\,du$.

$\Rightarrow F = 19{,}600 \int_2^0 u\left(44-(4-u^2)\right)(-2u\,du) = -39{,}200 \int_2^0 (40u^2 + u^3)\,du$.

$= 39{,}200 \left[\frac{40}{3}u^3 + \frac{1}{4}u^4\right]_0^2 = 39{,}200 \left[\frac{320}{3} + 4\right] = 4{,}338{,}133.33\ N.$

4. Sketch graph with direction of the following: (10 pts)

a) $\begin{cases} x = 2\cos(t)+1 \\ y = 3\sin(t)-2 \end{cases}$ (5 pts)

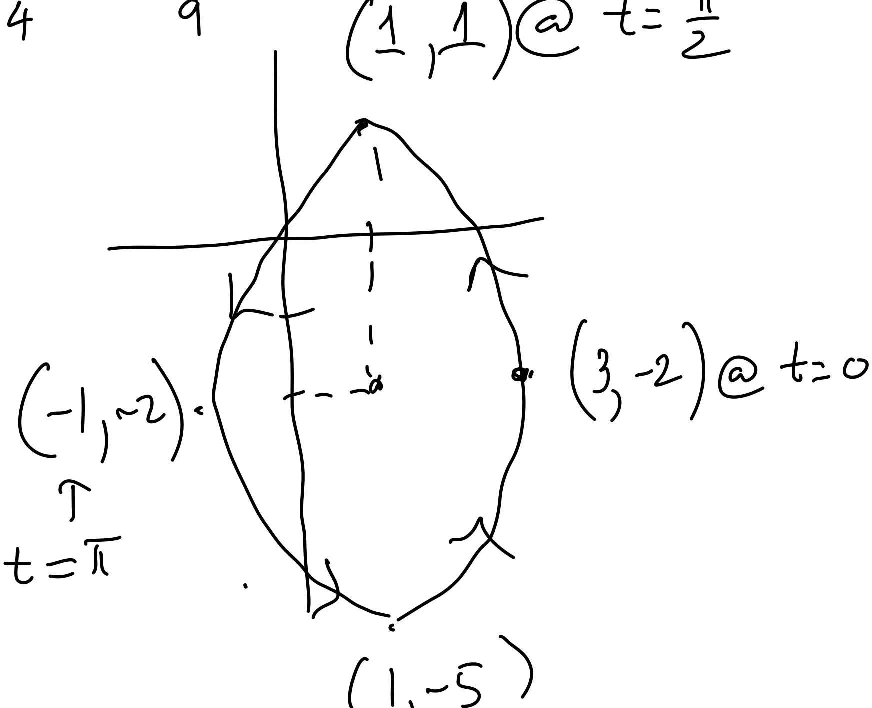
$$\begin{cases} \cos t = \frac{x-1}{2} \\ \sin t = \frac{y+2}{3} \end{cases} \Rightarrow \begin{array}{l} \cos^2 t = \frac{(x-1)^2}{4} \\ \sin^2 t = \frac{(y+2)^2}{9} \\ \hline 1 = \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} \end{array}$$

Elipse centered at $(1,-2)$

$$t = 0 \Rightarrow \begin{cases} x = 3 \\ y = -2 \end{cases}$$

$$t = \frac{\pi}{2} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$t = \pi \Rightarrow \begin{cases} x = -1 \\ y = -2 \end{cases}$$



b) $\begin{cases} x = 5\cos^2(t)-1 \\ y = 2\sin(t)+3 \end{cases}$ (5 pts)

$$\begin{array}{l} \cos^2 t = \frac{x+1}{5} \\ (\sin t)^2 = \left(\frac{y-3}{2}\right)^2 \\ \hline 1 = \frac{x+1}{5} + \frac{(y-3)^2}{4} \end{array}$$

$$\Rightarrow 5 = x+1+\frac{5}{4}(y-3)^2$$

$$x = -\frac{5}{4}(y-3)^2 + 4$$

parabola: vertex: $(4,3)$: $\supset$

we have: $0 \le \cos^2 t \le 1$

$0 \le 5\cos^2 t \le 5$

$-1 \le \cos^2 t - 1 \le 4$

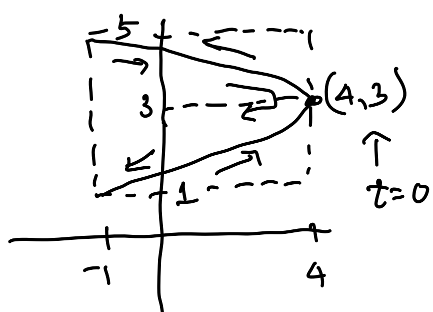
$\boxed{-1 \le x \le 4}$

and $-1 \le \sin t \le 1$

$-2 \le 2\sin t \le 2$

$1 \le 2\sin t + 3 \le 5$

$\boxed{1 \le y \le 5}$



$$t = 0 \Rightarrow \begin{cases} x = 4 \\ y = 3 \end{cases}$$

5. Determine equation of tangent line to $\begin{cases} x = \sqrt{3t+1} - 2 \\ y = 3\cos\left(\frac{\pi}{5}t\right) + t + 2 \end{cases}$; at $t = 5$ (5 pts)

$$\text{slope } m = y' = \frac{dy/dt}{dx/dt} = \left.\frac{-\frac{3\pi}{5}\sin\left(\frac{\pi}{5}t\right) + 1}{\frac{1}{2}(3t+1)^{-1/2} \cdot 3}\right|_{t=5} = \frac{-\frac{3\pi}{5}\sin(\pi) + 1}{\frac{1}{2}(16)^{-1/2} \cdot 3} = \frac{1}{3/8} = \frac{8}{3}$$

$$\text{points: } \left.\begin{cases} x_1 = \sqrt{3t+1} - 2 \\ y_1 = 3\cos\left(\frac{\pi}{5}t\right) + t + 2 \end{cases}\right|_{t=5} \Rightarrow \begin{array}{l} x_1 = \sqrt{16} - 2 = 2 \\ y_1 = 3\cos\pi + 5 + 2 = 4 \end{array}$$

$$\text{Eq}^{\underline{n}}: \quad y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{8}{3}(x - 2)$$

$$y = \frac{8}{3}x - \frac{16}{3} + 4$$

$$\boxed{y = \frac{8}{3}x - \frac{2}{3}}$$

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6. Determine point(s) on the curve: $\begin{cases} x = t^3 + 3t^2 - 9t + 1 \\ y = t^3 - 3t^2 - 24t - 1 \end{cases}$ where the tangent line is either horizontal or vertical. (5 pts)

$$\text{Slope: } m = y' = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 6t - 24}{3t^2 + 6t - 9} = \frac{3(t^2 - 2t - 8)}{3(t^2 + 2t - 3)} = \frac{t^2 - 2t - 8}{t^2 + 2t - 3},$$

Horizontal $\Rightarrow m = 0 \Rightarrow t^2 - 2t - 8 = 0 \Rightarrow (t-4)(t+2) = 0 \Rightarrow t = 4, -2.$

pts for horizontal tangent lines: $t = 4 \Rightarrow \begin{cases} x = 4^3 + 3(4^2) - 9(4) + 1 = 77 \\ y = 4^3 - 3(4^2) - 24(4) - 1 = -81 \end{cases}$ $(77, -81)$

$t = -2 \Rightarrow \begin{cases} x = (-2)^3 + 3(-2)^2 - 9(-2) + 1 = 23 \\ y = (-2)^3 - 3(-2)^2 - 24(-2) - 1 = 27 \end{cases}$ $(23, 27)$

Vertical $\Rightarrow m = \text{undefined} \Rightarrow t^2 + 2t - 3 = 0 \Rightarrow (t+3)(t-1) = 0 \Rightarrow t = -3, 1.$

pts for vertical tangent lines: $t = -3 \Rightarrow \begin{cases} x = (-3)^3 + 3(-3)^2 - 9(-3) + 1 = 28 \\ y = (-3)^3 - 3(-3)^2 - 24(-3) - 1 = 17 \end{cases}$

$(28, 17)$

$t = -1 \Rightarrow \begin{cases} x = (-1)^3 + 3(-1)^2 - 9(-1) + 1 = 12 \\ y = (-1)^3 - 3(-1) - 24(-1) - 1 = 25 \end{cases}$ $(12, 25)$

7. Determine the arc – length of the following curves: (10 pts)

a) $y=\frac{1}{4}x^2-\frac{1}{2}\ln(x)$ *for* $1\le x\le 2$

$L=\int ds=\int\sqrt{1+(y')^2}\,dx$ where $y'=\frac{1}{2}x-\frac{1}{2x}$

$\Rightarrow (y')^2=\left(\frac{1}{2}x-\frac{1}{2x}\right)^2=\frac{1}{4}x^2-\frac{1}{2}+\frac{1}{4x^2} \Rightarrow 1+(y')^2=\frac{1}{4}x^2+\frac{1}{2}+\frac{1}{4x^2}=\left(\frac{1}{2}x+\frac{1}{2x}\right)^2$

$\Rightarrow L=\int_1^2\sqrt{\left(\frac{1}{2}x+\frac{1}{2x}\right)^2}\,dx=\int_1^2\left(\frac{1}{2}x+\frac{1}{2x}\right)dx$

$=\frac{1}{4}x^2+\frac{1}{2}\ln|x|\Big|_1^2=1+\frac{1}{2}\ln x-\frac{1}{4}=\boxed{\frac{3}{4}+\ln\sqrt{2}}$

b) $\begin{cases}x=\cos^3(2t)\\ y=\sin^3(2t)\end{cases}$ *for* $0\le t\le\frac{\pi}{4}$ $\Rightarrow L=\int ds=\int_0^{\pi/4}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$\frac{dx}{dt}=-3\cos^2(2t)\sin(2t)\cdot 2=-6\cos^2(2t)\sin(2t)\Rightarrow\left(\frac{dx}{dt}\right)^2=36\cos^4(2t)\sin^2(2t)$

$\frac{dy}{dt}=3\sin^2(2t)\cos(2t)\cdot 2=6\sin^2(2t)\cos(2t)\Rightarrow\left(\frac{dy}{dt}\right)^2=36\sin^4(2t)\cos^2(2t)$

$\Rightarrow\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=36\cos^2(2t)\sin^2(2t)\left[\cos^2(2t)+\sin^2(2t)\right]=36\cos^2(2t)\sin^2(2t)$

$\Rightarrow L=\int_0^{\pi/4}\sqrt{36\cos^2(2t)\sin^2(2t)}\,dt=6\int_0^{\pi/4}\cos(2t)\sin(2t)\,dt$

Let $u=\sin(2t)$ $\begin{cases}t=\frac{\pi}{4}\Rightarrow u=\sin\left(\frac{\pi}{2}\right)=1\\ t=0\Rightarrow u=\sin(0)=0\end{cases}$ and $du=2\cos(2t)dt$, $\frac{du}{2}=\cos(2t)dt$

$\Rightarrow L=6\cdot\int_0^1 u\cdot\frac{du}{2}=3\int_0^1 u\,du=\frac{3}{2}\cdot u^2\Big|_0^1=\boxed{\frac{3}{2}}$

8. Find the areas of the surfaces generated by revolving the curved about the indicated axes. (10 pts)

a) $x = \cancel{\frac{1}{3}}y^4 + \frac{1}{8y^2}$ *for* $1 \le y \le 2$ *about the* $x-axis$ $\Rightarrow S = 2\pi \int r\,ds$ $\begin{cases} ds = \sqrt{1+(x')^2}\,dy \\ r = y. \end{cases}$

It should be: $x = \frac{1}{4}y^4 + \frac{1}{8y^2} = \frac{1}{4}y^4 + \frac{1}{8}y^{-2}$

$$x' = y^3 - \frac{1}{4}y^{-3} \Rightarrow (x')^2 = \left(y^3 - \frac{1}{4}y^{-3}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16}y^{-6}.$$

$$\Rightarrow 1+(x')^2 = y^6 + \frac{1}{2} + \frac{1}{16}y^{-6} = \left(y^3 + \frac{1}{4}y^{-3}\right)^2$$

$$\Rightarrow S = 2\pi \int_1^2 y\sqrt{\left(y^3+\frac{1}{4}y^{-3}\right)^2}\,dy = 2\pi \int_1^2 y\left(y^3+\frac{1}{4}y^{-3}\right)dy$$

$$= 2\pi \int_1^2 \left(y^4 + \frac{1}{4}y^{-2}\right)dy = 2\pi \left[\frac{1}{5}y^5 - \frac{1}{4y}\right]_1^2$$

$$= 2\pi\left[\frac{32}{5} - \frac{1}{20} - \frac{1}{5} + \frac{1}{4}\right] = \cdots = \#$$

b) $\begin{cases} x = e^t - t \\ y = 4e^{t/2} \end{cases}$; $0 \le t \le 1$; *about the* $x-axis$ $\begin{cases} \frac{dx}{dt} = e^t - 1 \Rightarrow \left(\frac{dx}{dt}\right)^2 = e^{2t} - 2e^t + 1 \\ \frac{dy}{dt} = 2e^{t/2} \Rightarrow \left(\frac{dy}{dt}\right)^2 = 4e^t. \end{cases}$

$S = 2\pi \int r\,ds$ $\begin{cases} ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ r = y = 4e^{t/2} \end{cases}$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2e^t + 1 = (e^t+1)^2.$$

$$\Rightarrow S = 2\pi \int_0^1 4e^{\frac{t}{2}} \cdot \sqrt{(e^t+1)^2}\,dt = 8\pi \int_0^1 e^{\frac{t}{2}}(e^t+1)\,dt$$

$$= 8\pi \int_0^1 \left(e^{\frac{3}{2}t} + e^{\frac{1}{2}t}\right)dt = 8\pi\left[\frac{2}{3}e^{\frac{3}{2}t} + 2e^{\frac{1}{2}t}\right]_0^1$$

$$= 8\pi\left[\frac{2}{3}e^{\frac{3}{2}} + 2e^{\frac{1}{2}} - \frac{2}{3} - 2\right] = \cdots = \#$$

9. Solve the following DE:

a) $\frac{dy}{dx}=\frac{e^{2x-y+1}}{e^{3y-x+2}}$; $y(0)=0$ (5 pts)

$$\frac{dy}{dx}=e^{2x-y+1-3y+x-2}=e^{3x-4y-1}=e^{3x-1}\cdot e^{-4y}$$

$$\frac{dy}{e^{-4y}}=e^{3x-1}dx \Rightarrow \int e^{4y}dy=\int e^{3x-1}dx$$

$$\Rightarrow \frac{1}{4}e^{4y}=\frac{1}{3}e^{3x-1}+C \quad ; y(0)=0$$

$$\Rightarrow \frac{1}{4}e^{0}=\frac{1}{3}e^{-1}+C \Rightarrow C=\frac{1}{4}-\frac{1}{3e}$$

$$\Rightarrow \boxed{\frac{1}{4}e^{4y}=\frac{1}{3}e^{3x-1}+\frac{1}{4}-\frac{1}{3e}}$$

b) $\frac{dy}{dx}=3x^2y^4-2xy^4-5y^4$; $y(0)=2$ (5 pts)

$$\frac{dy}{dx}=y^4(3x^2-2x-5)$$

$$\frac{dy}{y^4}=(3x^2-2x-5)dx$$

$$\int y^{-4}dy=\int(3x^2-2x-5)dx$$

$$\frac{y^{-3}}{-3}=x^3-x^2-5x+C.$$

$$\Rightarrow \frac{1}{y^3}=-3(x^3-x^2-5x)+C$$

$$y(0)=2 \Rightarrow \frac{1}{8}=C \Rightarrow y^3=\frac{1}{-3(x^3-x^2-5x)+\frac{1}{8}}$$

$$y=\sqrt[3]{\frac{1}{\frac{1}{8}-3x^3+3x^2+15x}}$$

10. A tank contains 500 gals of water with 10 lbs of salt initially. A solution containing ½ lb/gal of salt entering the tank at the rate of 2 gal / min, and the mixture is well stirred and pumped out at the same rate 2 gal/min. Set up a differential equation and then determine the concentration of salt in the tank after one hour. (10 pts)

Sol: Let $A(t)$ be the amount of salt in the tank at time t (in mins.)

$$\frac{dA}{dt} = \text{rate in} - \text{rate out} = \left(\frac{1}{2}\right)(2) - \frac{A}{500}\cdot 2 \; ; \quad A(0) = 10$$

$$\Rightarrow \frac{dA}{dt} = 1 - \frac{1}{250}A \Rightarrow \int \frac{dA}{1 - \frac{1}{250}A} = \int dt$$

$$-250 \ln\left|1 - \frac{1}{250}A\right| = t + C.$$

$$\ln\left|1 - \frac{1}{250}A\right| = -\frac{1}{250}t + C.$$

$$1 - \frac{1}{250}A = e^{-\frac{1}{250}t + C} = e^{-\frac{1}{250}t} \cdot \underbrace{\boxed{e^C}}_{=k} = k e^{-\frac{1}{250}t}$$

$$-250 + A = k e^{-\frac{1}{250}t} \Rightarrow A(t) = k e^{-\frac{1}{250}t} + 250.$$

$$A(0) = k \cdot e^0 + 250 = 10 \rightarrow k = -240.$$

$$A(t) = -240 e^{-\frac{1}{250}t} + 250.$$

One hour later $\Rightarrow A(60) = -240 e^{-\frac{1}{250}(60)} + 250.$

$$A(60) = 250 - 240 e^{-\frac{6}{25}} = 61.21 \text{ lbs of salt.}$$

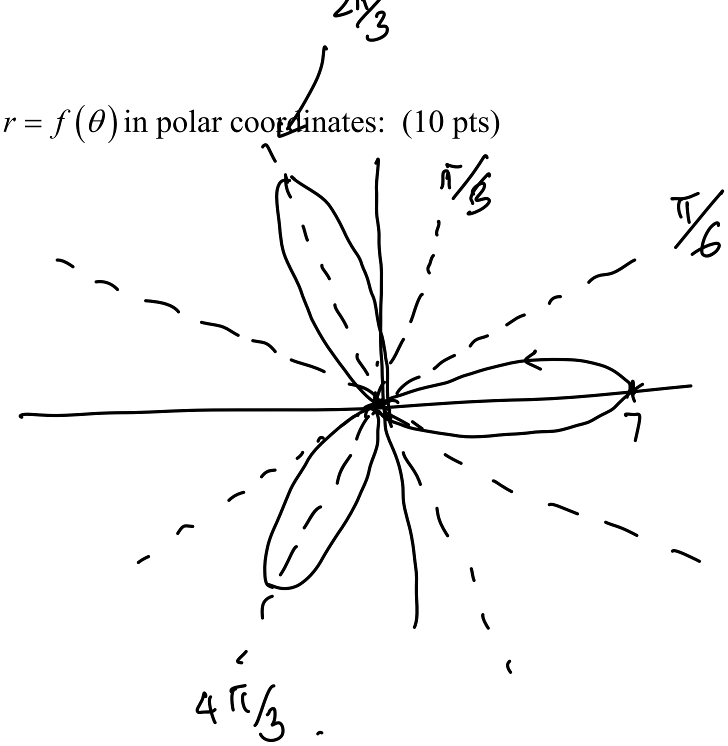
11. Sketch the graph of the following function $r = f(\theta)$ in polar coordinates: (10 pts)

a) $r = 7\cos(3\theta)$

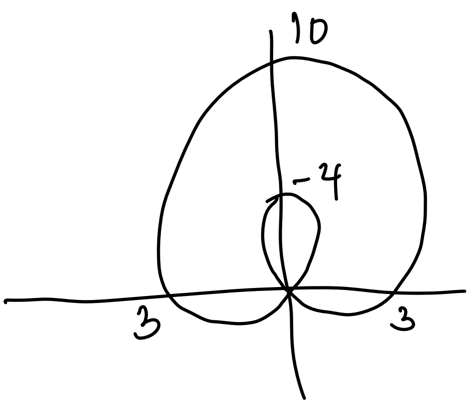
$\theta = 0 \Rightarrow r = 7$

$\theta = \frac{\pi}{6} \Rightarrow r = 7\cos\left(\frac{\pi}{2}\right) = 0$

$\theta = \frac{\pi}{3} \Rightarrow r = -7$

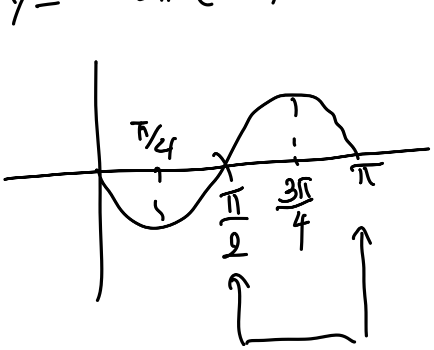


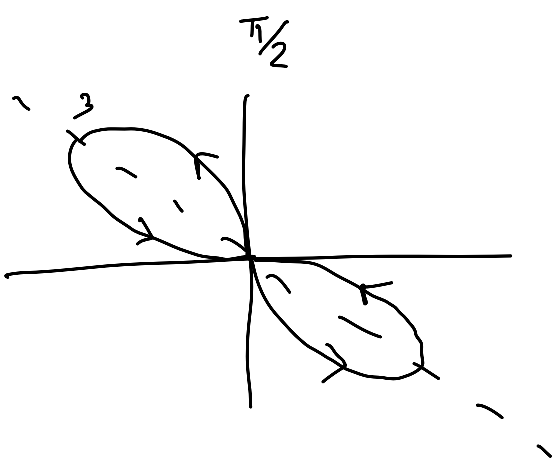
b) $r = 3 + 7\sin(\theta)$



c) $r^2 = -9\sin(2\theta) \Rightarrow r = \pm 3\sqrt{-\sin(2\theta)}$

$y = -\sin(2\theta)$





12. Find the slope to the curve $r = 2+2\cos(\theta)$ at $\theta = \frac{\pi}{4}$ (5 pts)

$$m = y' = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-2\sin\theta\cdot\sin\theta + (2+2\cos\theta)\cos\theta}{-2\sin\theta\cdot\cos\theta - (2+2\cos\theta)\sin\theta}.$$

$$= \frac{-2\sin^2\theta + 2\cos\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - 2\sin\theta - 2\cos\theta\sin\theta} = \left.\frac{\cos^2\theta - \sin^2\theta + \cos\theta}{-\sin\theta\cos\theta - \sin\theta}\right|_{\theta=\frac{\pi}{4}}$$

$$= \frac{\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right) + \cos\frac{\pi}{4}}{-\sin\frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}} = \frac{\left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{\sqrt{2}}{2}}{-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}}$$

$$= \frac{\sqrt{2}/2}{-\frac{1}{2} - \frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{-1-\sqrt{2}} = -\frac{\sqrt{2}}{1+\sqrt{2}}$$

13. ***Extra Credit:***

a) Using Calculus of surface area to prove that the surface area of a sphere of radius r is $S = 4\pi r^2$ (2 pts)

b) Given a curve $y = f(x)$ to be continuous and differentiable over the interval $[a,b]$. Derive that the arc – length of f(x) over $[a,b]$ is $L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$ (3 pts)