Exam #2

Summer 2021

Thursday 7/15/21

Math 181 – Calculus II

Name: KEY

Score:________________

Rules for taking exam:

1. Please put away all lecture notes, textbooks, or any electronic devices.
   (Except you are using iPad to take exam)
2. Only scientific calculators are allowed.
3. Scan all your work as one pdf file and submit it thru canvas.
   (Put the problem numbers in order and in the right direction (No side way or upside down). Double check if it includes all the pages that you want to submit, check your pdf file if it's readable, make sure there's no shade, shadow or black out that causes unreadable.)
4. All late submitted papers will be penalized 25% of your scores.

1. Test for convergence / divergence: (15 pts)

a) $\int_1^{\infty} \frac{\sqrt[3]{2x^8+3x^5+1}}{3x^2+4x+5}\,dx \geq \int_1^{\infty} \frac{\sqrt[3]{x^8}}{3x^2+4x^2+5x^2}\,dx = \int_1^{\infty} \frac{x^{8/3}}{12x^2}\,dx = \frac{1}{12}\int_1^{\infty} \frac{1}{x^{2-\frac{8}{3}}}\,\cancel{dx}$

Dominant term: $\frac{\sqrt[3]{x^8}}{x^2} = \frac{1}{x^{2-\frac{8}{3}}} = \frac{1}{x^{-2/3}}$

$= \frac{1}{12}\int_1^{\infty} \frac{1}{x^{-2/3}}\,dx$ $\left\{ \begin{array}{l} p = -\frac{2}{3} < 1 \text{ is} \\ \text{divergent by} \\ p-\text{Test} \end{array} \right.$

$\therefore$ by C.T.T. $\int_1^{\infty} f(x)\,dx$ is divergent

b) $\int_2^{\infty} \frac{7x^2+3x+1}{\sqrt{4x^7+3x^5+2}}\,dx \leq \int_2^{\infty} \frac{7x^2+3x^2+x^2}{\sqrt{x^7}}\,dx = \int_2^{\infty} \frac{11x^2}{x^{7/2}}\,dx$

Dominant terms: $\frac{x^2}{\sqrt{x^7}} = \frac{1}{x^{\frac{7}{2}-2}} = \frac{1}{x^{3/2}}$

$\Rightarrow 11\int_2^{\infty} \frac{1}{x^{\frac{7}{2}-2}}\,dx = 11\int_2^{\infty} \frac{1}{x^{3/2>1}}\,dx$ $\left\{ \begin{array}{l} p = \frac{3}{2} > 1 \text{ is} \\ \text{convergent by} \\ p-\text{Test} \end{array} \right.$

$\therefore$ by C.T.T. $\int_2^{\infty} f(x)\,dx$ is convergent

c) $\int_0^{\pi} \frac{1}{x^3\cos^2(3x)\,\cancel{dx}}\,dx$

for $0 \leq x \leq \pi \Longrightarrow$ we have: $0 \leq \cos^2(3x) \leq 1$

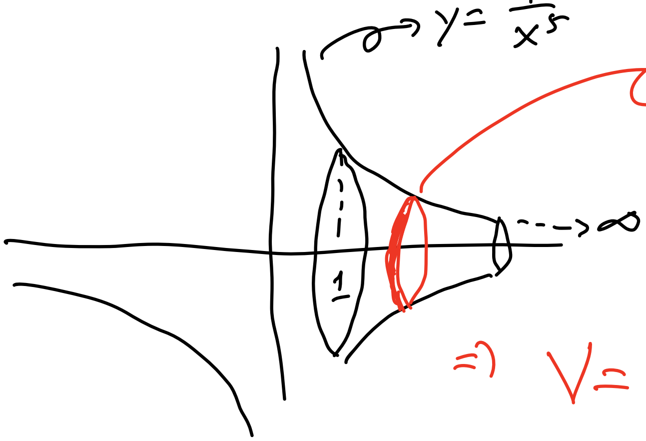
$\Rightarrow 0 \leq x^3\cos^2(3x) \leq x^3$

$\Rightarrow \frac{1}{x^3\cos^2(3x)} \geq \frac{1}{x^3} \Rightarrow \int_0^{\pi} \frac{1}{x^3\cos^2(3x)}\,dx \geq \int_0^{\pi} \frac{1}{x^3}\,dx = \lim_{t\to 0^+}\int_t^{\pi} x^{-3}\,dx$

$= \lim_{t\to 0^+} \frac{x^{-2}}{-2}\Big|_t^{\pi} = -\frac{1}{2}\lim_{t\to 0^+}\left(\frac{1}{\pi^2} - \frac{1}{t^2}\right) = \infty$

$\therefore$ by C.T.T. $\int_0^{\pi} \frac{1}{x^3\cos^2(3x)}\,dx$ is divergent

2. The region bounded by $y = \frac{1}{x^5}$ *and* $y = 0$ *for* $x \geq 1$ is rotated about the x – axis. Calculate its volume. (5 pts)



$$V = \pi \cdot r^2 \cdot dx$$

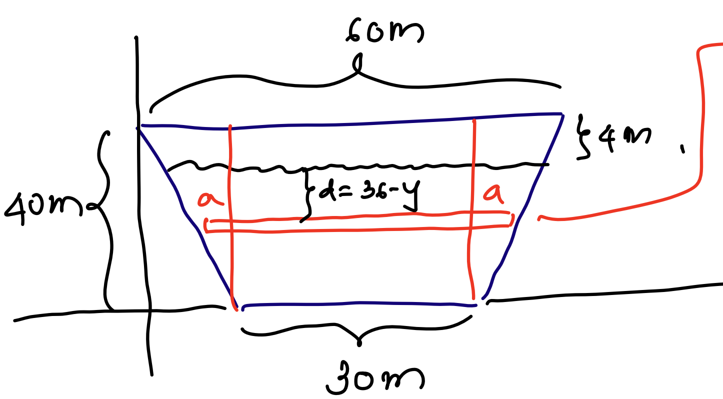
where r is $= \frac{1}{x^5} - 0 = \frac{1}{x^5}$

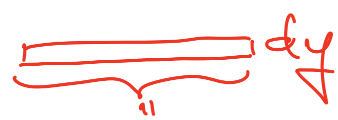
$$\Rightarrow V = \pi \int_1^{\infty} \left(\frac{1}{x^5}\right)^2 dx = \pi \lim_{t \to \infty} \int_1^t x^{-10}\, dx$$

$$= \pi \lim_{t \to \infty} \frac{x^{-9}}{-9} \Big|_1^t = -\frac{\pi}{9} \lim_{t \to \infty} \left[\frac{1}{t^9} - 1\right] = \frac{\pi}{9}$$

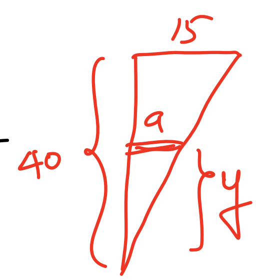
Convergent.

3. A dam has the shape of a trapezoid. The height is 40 m, and the width is 60 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam. (5 pts)





$L = 30 + 2a$.



similar Δ : $\frac{a}{15} = \frac{y}{20}$

$\Rightarrow a = \frac{5}{4} y$.

$\Rightarrow L = 30 + 2\left(\frac{5}{4} y\right) = 30 + \frac{5}{2} y$

$A = \left(30 + \frac{5}{2} y\right) dy$

$d = (36 - y)$

$V = \left(30 + \frac{5}{2} y\right)(36 - y)\, dy$

$F = 9800 \int_0^{36} \left(30 + \frac{5}{2} y\right)(36 - y)\, dy$.

$= 9800 \int_0^{36} \left[1080 + 60y - \frac{5}{2} y^2\right] dy$

$9800 \left[1080y + 30y^2 - \frac{5}{6} y^3\right]_0^{36}$

$= 9800 \left[1080(36) + 30(36)^2 - \frac{5}{6}(36)^3\right]$.

$= 38{,}024{,}000$ N.

4. Sketch graph with direction of the following: (10 pts)

a) $\begin{cases} x = 2\cos^2(2t)+1 \\ y = 3\sin(2t)-2 \end{cases}$; $t \in R$ (5 pts)

$\cos^2(2t) = \dfrac{x-1}{2}$

$(\sin(2t))^2 = \left(\dfrac{y+2}{3}\right)^2$

$1 = \dfrac{x-1}{2} + \dfrac{(y+2)^2}{9}$

$2 = x - 1 + \frac{2}{9}(y+2)^2$

$X = -\frac{2}{9}(y+2)^2 + 3$

Parabola: vertex: $(3,-2)$

We have: $0 \le \cos^2(2t) \le 1$

$0 \le 2\cos^2(2t) \le 2$

$1 \le 2\cos^2(2t)+1 \le 3$

$\Rightarrow$ $\boxed{1 \le x \le 3}$

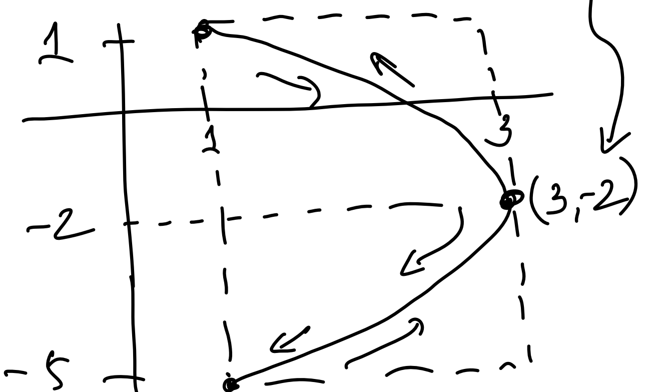
$-1 \le \sin(2t) \le 1$

$-3 \le 3\sin(2t) \le 3$

$-5 \le 3\sin(2t) - 2 \le 1$

$\boxed{-5 \le y \le 1}$

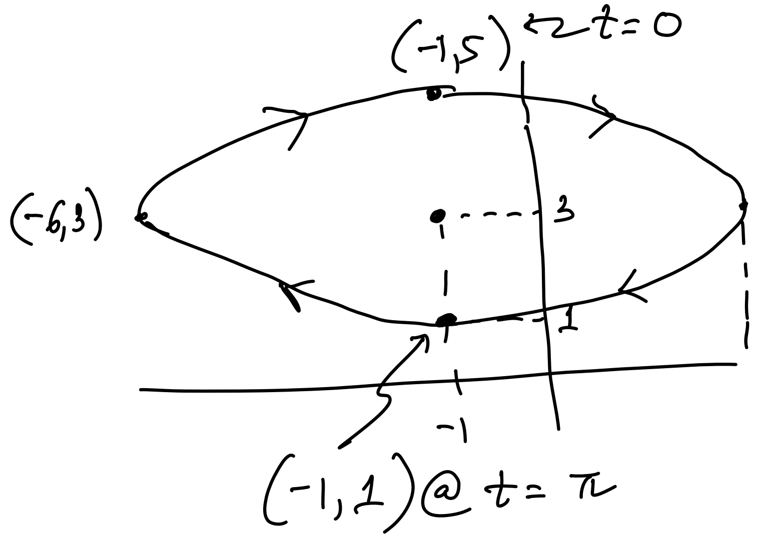
$t=0 \begin{cases} x = 3 \\ y = -2 \end{cases}$



b) $\begin{cases} x = 5\sin(t)-1 \\ y = 2\cos(t)+3 \end{cases}$ (5 pts) $\Rightarrow$ $\begin{cases} \sin(t) = \dfrac{x+1}{5} \\ \cos(t) = \dfrac{y-3}{2} \end{cases}$ $\Rightarrow$ $\begin{cases} \sin^2 t = \dfrac{(x+1)^2}{25} \\ + \\ (\cos^2 t) = \dfrac{(y-3)^2}{4} \end{cases}$

$1 = \dfrac{(x+1)^2}{25} + \dfrac{(y-3)^2}{4}$

Elipse centred at $(-1,3)$



$t=0 \Rightarrow \begin{cases} x = -1 \\ y = 5 \end{cases}$

$t = \frac{\pi}{2} \Rightarrow \begin{cases} x = 4 \\ y = 3 \end{cases}$

$t = \pi \Rightarrow \begin{cases} x = -1 \\ y = 1 \end{cases}$

5. Determine equation of tangent line to $\begin{cases} x = 2t - 3\sin\left(\frac{\pi}{3}t\right) \\ y = 3\sqrt[3]{t^2-1} - 2 \end{cases}$; $at\ t = 3$ (5 pts)

$$\text{slope } m = y' = \frac{dy/dt}{dx/dt} = \frac{(t^2-1)^{-2/3}(2t)}{2 - \pi\cos\left(\frac{\pi}{3}t\right)}\Bigg|_{t=3} = \frac{8^{-2/3} \cdot 6}{2 - \pi\cos\pi} = \frac{3/2}{2+\pi} = \frac{3}{2(2+\pi)}$$

$$\text{point: } \begin{cases} x_1 = 2t - 3\sin\left(\frac{\pi}{3}t\right) \\ y_1 = 3\sqrt[3]{t^2-1} - 2 \end{cases}\Bigg|_{t=3} = \begin{cases} x_1 = 6 - 3\sin\pi = 6 \\ y_1 = 3\sqrt[3]{8} - 2 = 4 \end{cases}$$

$$\text{sol}: \quad y - y_1 = m(x - x_1)$$

$$\boxed{y - 4 = \frac{3}{2(2+\pi)} \cdot (x - 6)}$$

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6. Determine point(s) on the curve: $\begin{cases} x = t^3 + 3t^2 - 9t + 1 \\ y = t^3 - 3t^2 - 24t + 2 \end{cases}$ where the tangent line is either horizontal or vertical. (5 pts)

$$\text{Slope: } m = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 6t - 24}{3t^2 + 6t - 9} = \frac{3(t^2 - 2t - 8)}{3(t^2 + 2t - 3)} = \frac{(t-4)(t+2)}{(t+3)(t-1)}$$

Horizontal $\Rightarrow m = 0 \Rightarrow (t-4)(t+2) = 0 \Rightarrow t = 4, -2.$

at $t = 4 \Rightarrow \begin{cases} x = (4)^3 + 3(4)^2 - 9(4) + 1 = 77 \\ y = (4)^3 - 3(4)^2 - 24(4) + 2 = -78 \end{cases}$; $(77, -78)$ & $t = -2$ $\begin{cases} x = (-2)^3 + 3(-2)^2 - 9(-2) + 1 = 23 \\ y = (-2)^3 - 3(-2)^2 - 24(-2) + 2 = 30 \end{cases}$ $(23, 30)$

Vertical $\Rightarrow m =$ undefined $\Rightarrow (t+3)(t-1) = 0 \Rightarrow t = -3, 1.$

$t = -3 \Rightarrow \begin{cases} x = (-3)^3 + 3(-3)^2 - 9(-3) + 1 = 28 \\ y = (-3)^3 - 3(-3)^2 - 24(-3) + 2 = 20 \end{cases}$; $(28, 20)$ & $t = 1 \Rightarrow \begin{cases} x = 1^3 + 3(1) - 9(1) + 1 = -4 \\ y = 1^3 - 3(1) - 24(1) + 2 = -24 \end{cases}$

$(-4, -24)$

7. Determine the arc – length of the following curves: (10 pts) 2

a) $x = \frac{1}{12}y^3 + \frac{1}{y}$ *for* $1 \le y \le 2$ $\Rightarrow L = \int ds = \int_1^2 \sqrt{1 + (x')^2}\, dy$

$$x' = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow (x')^2 = \left(\frac{1}{4}y^2 - \frac{1}{y^2}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}$$

$$\Rightarrow 1 + (x')^2 = \frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4} = \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2.$$

$$\Rightarrow L = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + y^{-2}\right) dy$$

$$= \frac{1}{12}y^3 - \frac{1}{y}\Big|_1^2 = \frac{2}{3} - \frac{1}{2} - \frac{1}{12} + 1 = \boxed{\#}$$

b) $\begin{cases} x = e^{3t}\sin(t) \\ y = e^{3t}\cos(t) \end{cases}$ *for* $0 \le t \le \pi$ $\Rightarrow L = \int ds = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\frac{dx}{dt} = e^{3t}(3\sin t + \cos t) \Rightarrow \left(\frac{dx}{dt}\right)^2 = e^{6t}\left[9\sin^2 t + 6\sin t\cos t + \cos^2 t\right]$$

$$\frac{dy}{dt} = e^{3t}(3\cos t - \sin t) \Rightarrow \left(\frac{dy}{dt}\right)^2 = e^{6t}\left[9\cos^2 t - 6\sin t\cos t + \sin^2 t\right]$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{6t}\left[9 + 1\right] = 10e^{6t}.$$

$$\Rightarrow L = \int_0^\pi \sqrt{10e^{6t}}\, dt = \sqrt{10}\int_0^\pi e^{3t}\, dt = \frac{\sqrt{10}}{3}e^{3t}\Big|_0^\pi.$$

$$\Rightarrow L = \frac{\sqrt{10}}{3}\left[e^{3\pi} - 1\right], = \dots = \#$$

8. Find the areas of the surfaces generated by revolving the curved about the indicated axes. (10 pts)

a) $x = y^2 - \frac{1}{8}\ln(y)$ *for* $1 \le y \le 2$ *about the* $x-axis$

$$S = 2\pi \int r\,ds \text{ where } \begin{cases} ds = \sqrt{1+(x')^2}\,dy \\ r = y \end{cases} \quad \text{match.}$$

$$x' = 2y - \frac{1}{8y} \Rightarrow (x')^2 = 4y^2 - \frac{1}{2} + \frac{1}{64y^2} \Rightarrow 1+(x')^2 = \left(2y + \frac{1}{8y}\right)^2$$

$$\Rightarrow S = 2\pi \int_1^2 y\sqrt{\left(2y+\frac{1}{8y}\right)^2}\,dy = 2\pi \int_1^2 y\left(2y+\frac{1}{8y}\right)dy$$

$$= 2\pi \int_1^2 \left(2y^2 + \frac{1}{8}\right)dy = 2\pi\left[\frac{2}{3}y^3 + \frac{1}{8}y\right]_1^2$$

$$= 2\pi\left[\frac{16}{3} + \frac{1}{4} - \frac{2}{3} - \frac{1}{8}\right] = \dots = \#$$

b) $\begin{cases} x = e^t - t \\ y = 4e^{t/2} \end{cases}$; $0 \le t \le 1$; *about the* $x-axis$ $\Rightarrow S = 2\pi\int r\,ds$ where $\begin{cases} ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ r = y = 4e^{t/2} \end{cases}$

$$\frac{dx}{dt} = e^t - 1 \Rightarrow \left(\frac{dx}{dt}\right)^2 = e^{2t} - 2e^t + 1$$

$$\frac{dy}{dt} = 2e^{t/2} \Rightarrow \left(\frac{dy}{dt}\right)^2 = 4e^t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} + 2e^t + 1 = (e^t+1)^2.$$

$$S = 2\pi \int_0^1 4e^{t/2}\sqrt{(e^t+1)^2}\,dt = 8\pi \int_0^1 e^{t/2}(e^t+1)\,dt$$

$$= 8\pi \int_0^1 \left(e^{\frac{3}{2}t} + e^{\frac{1}{2}t}\right)dt = 8\pi\left[\frac{2}{3}e^{\frac{3}{2}t} + 2e^{\frac{1}{2}t}\right]_0^1$$

$$= 8\pi\left[\frac{2}{3}e^{\frac{3}{2}} + 2e^{\frac{1}{2}} - \frac{3}{2} - 2\right] = \dots = \#$$

9. Solve the following DE:

a) $\frac{dy}{dx} = x^3y^2 - 5xy^2 + 3y^2;\ y(0)=1$ (5 pts)

$$\frac{dy}{dx} = y^2(x^3 - 5x + 3) \Rightarrow \frac{dy}{y^2} = (x^3 - 5x + 3)dx$$

$$\Rightarrow \int y^{-2}dy = \int (x^3 - 5x + 3)dx \Rightarrow \frac{y^{-1}}{-1} = \frac{1}{4}x^4 - \frac{5}{2}x^2 + 3x + C$$

$$y(0)=1 \begin{cases} x=0 \\ y=1 \end{cases} \Rightarrow -1 = C \Rightarrow C=-1$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{4}x^4 - \frac{5}{2}x^2 + 3x - 1 \Rightarrow y = \frac{-1}{\frac{1}{4}x^4 - \frac{5}{2}x^2 + 3x - 1}$$

b) $\frac{dy}{dx} = \frac{e^{3-2x+y}}{e^{3y-x+1}}$ (5 pts)

$$\frac{dy}{dx} = e^{3-2x+y-3y+x-1} = e^{-x-2y+2}$$

$$\frac{dy}{dx} = e^{-x+2} \cdot e^{-2y} \Rightarrow \frac{dy}{e^{-2y}} = e^{-x+2}dx$$

$$\Rightarrow \int e^{2y}dy = \int e^{-x+2}dx$$

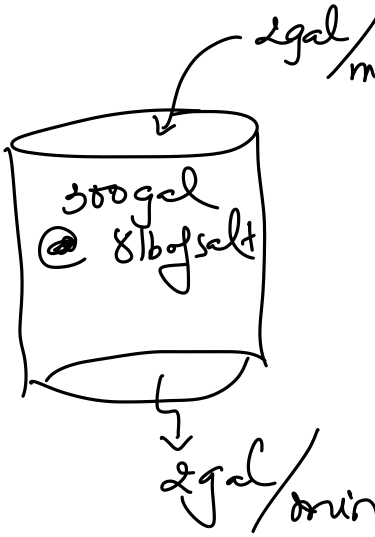
$$\frac{1}{2}e^{2y} = -e^{-x+2} + C$$

$$e^{2y} = -2e^{-x+2} + C$$

$$\Rightarrow \text{Sol}^n: \ y = \frac{1}{2}\ln\left(C - 2e^{-x+2}\right).$$

10. A tank contains 300 gals of water with 8 lbs of salt initially. A solution containing ½ lb/gal of salt entering the tank at the rate of 2 gal / min, and the mixture is well stirred and pumped out at the same rate 2 gal/min. Set up a differential equation and then determine the concentration of salt in the tank after 30 minutes. (10 pts)

Sol: Let $A(t)$ be the amount of salt in the tank at time t. (in mins.)



$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

$$= 2\left(\frac{1}{2}\right) - \frac{A}{300}\cdot 2 \;; \quad A(0)=8$$

$$\frac{dA}{dt} = 1 - \frac{1}{150}A .$$

$$\Rightarrow \int \frac{dA}{1-\frac{1}{150}A} = \int dt \Rightarrow -150\ln\left|1-\frac{1}{150}A\right| = t + C$$

$$\ln\left|1-\frac{1}{150}A\right| = -\frac{1}{150}t + C \Rightarrow 1-\frac{1}{150}A = e^{-\frac{1}{150}t+C} = ke^{-\frac{1}{150}t} .$$

$$\Rightarrow -150 + A = ke^{-\frac{1}{150}t} \Rightarrow A(t) = 150 + ke^{-\frac{1}{150}t} .$$

$$A(0) = 150 + k = 8 \Rightarrow k = -142 .$$

$$\Rightarrow A(t) = 150 - 142e^{-\frac{1}{150}t} .$$

After 30 mins $\Rightarrow A(30) = 150 - 142e^{-\frac{1}{150}(30)}$

$$\underline{A(30) = 33.74 \text{ lbs} .}$$

11. Sketch the graph of the following function $r = f(\theta)$ in polar coordinates: (10 pts)

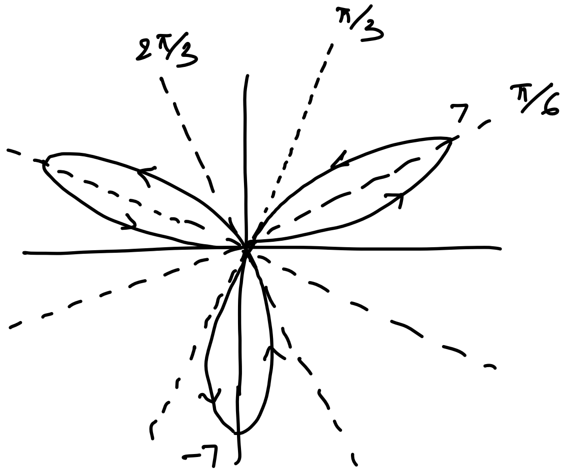
a) $r = 7\sin(3\theta)$

$\theta = 0 \Rightarrow r = 7\sin(0) = 0$

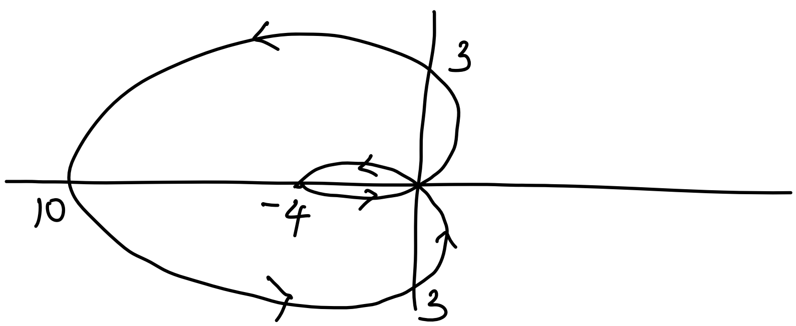
$\theta = \frac{\pi}{6} \Rightarrow r = 7\sin(\frac{\pi}{2}) = 7$

$\theta = \frac{\pi}{3} \Rightarrow r = 7\sin(\pi) = 0$

$\theta = \frac{\pi}{2} \Rightarrow r = 7\sin(\frac{3\pi}{2}) = -7$

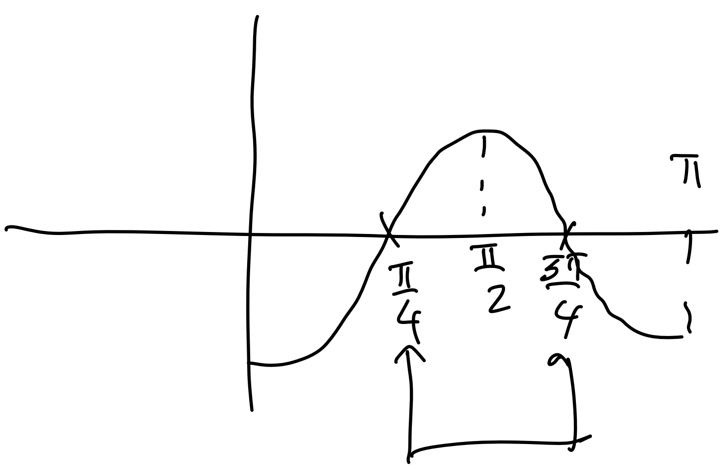


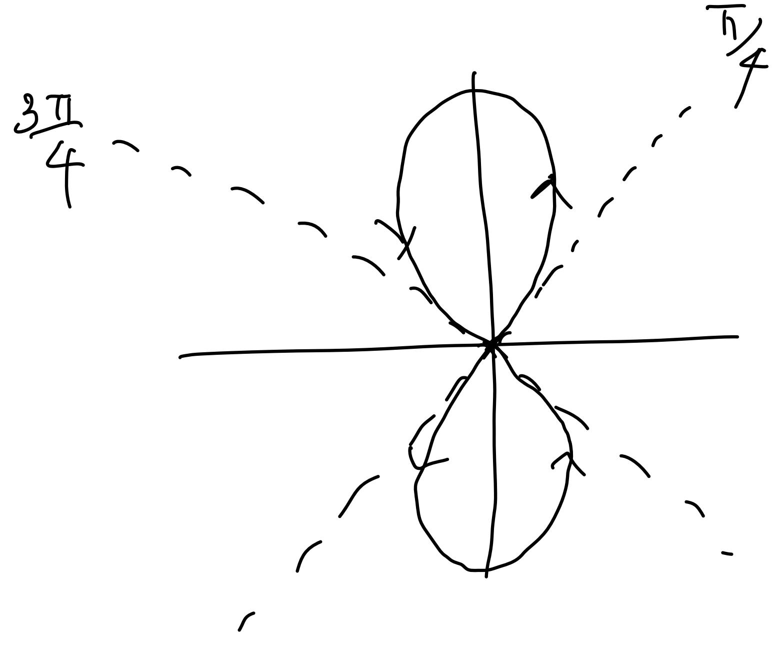
b) $r = 3 - 7\cos(\theta)$



c) $r^2 = -25\cos(2\theta) \Rightarrow r = \pm\sqrt{-25\cos(2\theta)} = \pm 5\sqrt{-\cos(2\theta)}$

$y = -\cos(2\theta)$





12. Find the slope to the curve $r = 2 - 2\sin(\theta)$ at $\theta = \frac{\pi}{4}$ (5 pts)

$$\text{slope } y' = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-2\cos\theta\sin\theta + (2-2\sin\theta)\cos\theta}{-2\cos\theta\cos\theta - (2-2\sin\theta)\sin\theta}$$

$$= \frac{-2\left[\cos\theta\sin\theta - \cos\theta + \cos\theta\sin\theta\right]}{-2\left[\cos^2\theta + \sin\theta - \sin^2\theta\right]} = \frac{\overbrace{2\cos\theta\sin\theta}^{\sin(2\theta)} - \cos\theta}{\cos(2\theta) + \sin\theta}.$$

$$m = y'\Big|_{\theta=\frac{\pi}{4}} = \frac{\sin\left(\frac{\pi}{2}\right) - \cos\frac{\pi}{4}}{\cos\left(\frac{\pi}{2}\right) + \sin\frac{\pi}{4}} = \frac{1 - \frac{\sqrt{2}}{2}}{0 + \frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} - 1 = \boxed{\sqrt{2} - 1}$$

13. ***Extra Credit:***

a) Using Calculus of surface area to prove that the surface area of a sphere of radius r is $S = 4\pi r^2$ (2 pts)

b) Given a curve $y = f(x)$ to be continuous and differentiable over the interval $[a,b]$. Derive that the arc – length of f(x) over $[a,b]$ is $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$ (3 pts)