1. Solve by REF.

a) $\begin{cases} 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 - 2x_3 = -5 \end{cases} \Rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \\ \downarrow & \downarrow & \downarrow \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right] \end{array}$

$\xrightarrow{R_1 \Leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right]$

$\xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$\hookrightarrow x_3 = 2.$

~~seen~~ for $R_2 \Rightarrow x_2 + x_3 = 3 \Rightarrow x_2 + 2 = 3 \Rightarrow x_2 = 1$

for $R_1 \Rightarrow x_1 - x_2 - 2\cancel{x_3} = -5$

$x_1 - 1 - 2(2) = -5 \Rightarrow x_1 = 0$

Solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

1b) $\begin{cases} x_1 - x_2 - x_3 + 2x_4 = 1 \\ 2x_1 - 2x_2 - x_3 + 3x_4 = 3 \\ -x_1 + x_2 - x_3 = -3 \end{cases}$

$$\Rightarrow \begin{array}{c} \begin{array}{cccc} x_1 \downarrow & x_2 \downarrow & x_3 \downarrow & x_4 \downarrow \end{array} \\ \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \end{array} \xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & -2 \end{array}\right]$$

$$2x_4 = -2 \Rightarrow x_4 = -1$$

for $R_2 \Rightarrow x_3 + x_4 = 1 \Rightarrow x_3 - 1 = 1 \Rightarrow x_3 = 2.$

$R_1 \Rightarrow x_1 \cancel{+} x_2 - x_3 + x_4 = 1$; let $x_2 = t$

$x_1 - t - 2 - 1 = 1 \Rightarrow x_1 = t + 4$

Solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t+4 \\ t \\ 2 \\ -1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 2 \\ -1 \end{bmatrix}$

2. For what value(s) of k, if any, will the system have.

a) No solution ; b) a unique solution, ~~I~~

c) Infinitely many solutions.

$$\begin{cases} x + y + kz = 1 \\ x + ky + z = 1 \\ kx + y + z = -2 \end{cases}$$

Sol:
$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array}\right] \xrightarrow[-kR_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -2-k \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & -2-k \end{array}\right]$$

$$(2-k-k^2)z = -2-k.$$

$$\Rightarrow z = \frac{-2-k}{2-k-k^2}$$

$$\Rightarrow z = \frac{-(2+k)}{(2+k)(1-k)}$$

for ⟨a⟩ No solution $\Rightarrow (2+k) \neq 0 \Rightarrow z = \dfrac{1}{1-k} \Rightarrow \boxed{k=1}$

⟨b⟩ a unique solution $\Longrightarrow \boxed{k \neq 2,\ k \neq 1.}$

⟨c⟩ Infinitely many solution $\Longrightarrow \boxed{k = 2}$

since $z \cdot 0 = 0 \Rightarrow z = t.$

3. Given $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 4 & -1 \\ 3 & 1 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 7 & -5 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

Perform the following, if possible, AB, BA, $B^T A$, B^2, C^2

$$A \cdot B = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 4 & -1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} 15 & -11 \\ -1 & 1 \\ 26 & -19 \end{bmatrix}$$

$-1.$

$B_{3\times 2} \cdot A_{3\times 3}$: Not possible.

$\underbrace{2 \neq 3}$

$$B^T \cdot A = \begin{bmatrix} 1 & 2 & 7 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -2 & 4 & -1 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 & 20 \\ -12 & -13 & -15 \end{bmatrix}$$

$B^2 = B_{2\times 3} \cdot B_{2\times 3}$: not possible.

$\underbrace{3 \neq 2}$

$$C^2 = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ -3 & -2 \end{bmatrix}$$

4) a) Let A be an $n \times n$ matrix. Prove that $A - A^T$ is a skew-symmetric matrix.

Proof: By definition a matrix B is ~~sym~~ a skew-symmetric when: $B^T = -B$.

$\Rightarrow$ Need to show: $(A - A^T)^T = -(A - A^T)$

since $(A+B)^T = A^T + B^T \Rightarrow$ $(A - A^T)^T = A^T - (A^T)^T$.

since $(A^T)^T = A \Rightarrow$ $A^T - (A^T)^T = A^T - A$.

since $\alpha(A+B) = \alpha A + \alpha B \Rightarrow$ $A^T - A = -[-A^T] - [A]$

$= -[-A^T + A]$

since $A + B = B + A$ $\Rightarrow$ $-(-A^T + A) = -(A - A^T)$

$$\Rightarrow (A - A^T)^T = -(A - A^T)$$

By def. $\Rightarrow$ $A - A^T$ is a skew-symmetric matrix.

4b) let A and B be $n \times n$ symmetric matrices
Prove that the sum $A+B$ is also symmetric

Proof: By def. A & B are symmetric

$$\Rightarrow A^T = A \quad \text{and} \quad B^T = B.$$

$\Rightarrow$ We need to show: $(A+B)^T = (A+B)$

Since $(A+B)^T = A^T + B^T$.

Since $A^T = A$ & $B^T = B \Rightarrow (A+B)^T = A+B$.

$\Rightarrow A+B$ is also a symmetric matrix.

5> Determine the inverse matrix of the following:

a) $$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} = \left[A \mid I\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \Leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \\ 0 & -2 & 6 & -2 & 1 & 0 \end{array}\right]$$

Bottom up

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \\ 0 & 0 & -2 & -4 & 1 & 2 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3+R_1]{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -2 & -4 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 3/2 & 1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -2 & -4 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 3/2 & 1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1/2 & -1 \end{array}\right]$$

$\Rightarrow$ The inverse matrix $A^{-1} = \begin{bmatrix} -3 & 3/2 & 1 \\ 3 & -1 & -1 \\ 2 & -1/2 & -1 \end{bmatrix}$

$\Rightarrow$ A is a non-singular matrix.

5b)

$$B = \begin{bmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}$$

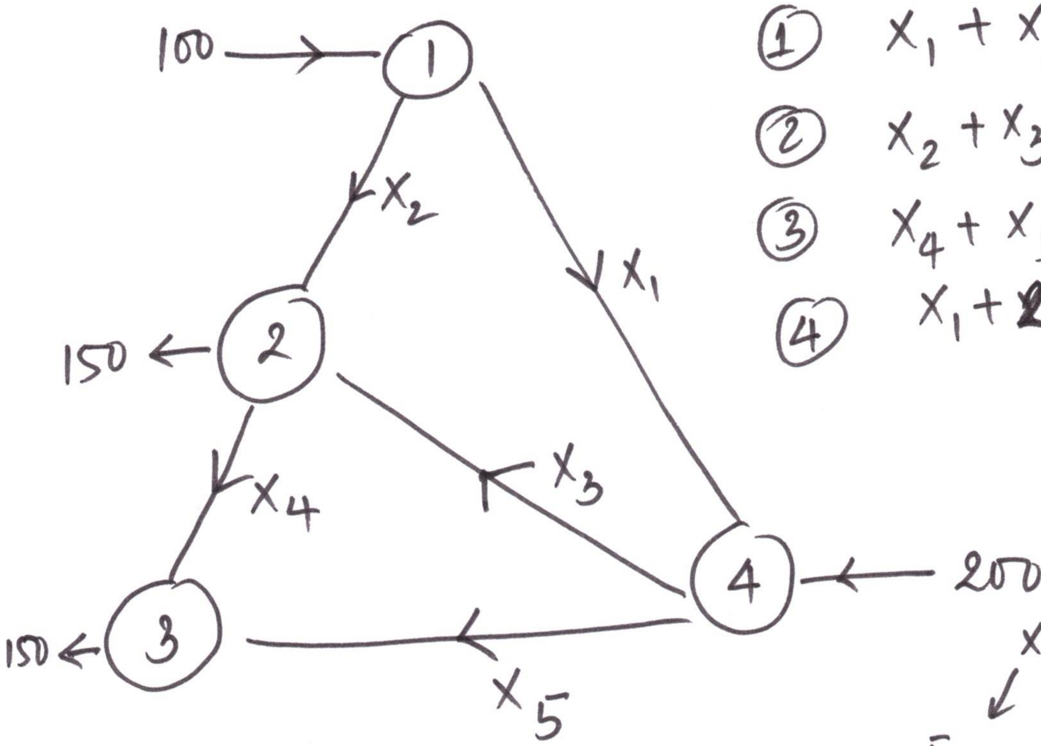
$$\Rightarrow [B \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ -4 & -1 & 6 & 0 & 1 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 + R_3]{2R_1 + R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_2 + R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & -5 & -3 & 1 \end{array}\right]$$

B is a singular matrix

it has no inverse.



① $x_1 + x_2 = 100$

② $x_2 + x_3 = x_4 + 150$

③ $x_4 + x_5 = 150$

④ $x_1 + 200 = x_3 + x_5$

$$\Rightarrow \begin{cases} x_1 + x_2 = 100 \\ x_2 + x_3 - x_4 = 150 \\ x_4 + x_5 = 150 \\ x_1 - x_3 - x_5 = -200 \end{cases} \Longrightarrow \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 1 & 0 & -1 & 0 & -1 & -200 \end{array}$$

$$\xrightarrow{-R_1+R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & -1 & -1 & 0 & -1 & -300 \end{array}\right] \xrightarrow{R_2+R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & -1 & -1 & -150 \end{array}\right]$$

$$\xrightarrow{R_3+R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 + x_5 = 150 \Rightarrow x_4 = 150 - t$

$x_5 = t$.

$x_2 + x_3 - x_4 = 150$

$x_2 + s - (150 - t) = 150$

$x_3 = s$

$x_2 = -s - t + 300$

and $x_1 + x_2 = 100$

$x_1 - s - t + 300 = 100$

$x_1 = s + t - 200$

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} s+t-200 \\ -s-t+300 \\ s \\ 150-t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -200 \\ 300 \\ 0 \\ 150 \\ 0 \end{bmatrix}$$