Exam #1 ***Math 260*** ***9/17/20*** ***Name:*** Key

Show all your work clearly. No Work, No Credit.

1. Solve by using REF: (15pts)

a) $$\begin{cases} 3x_1 - 2x_3 + x_4 = -2 \\ 2x_1 + 3x_2 - 2x_3 = -3 \\ x_1 - 3x_2 + x_4 = 1 \\ 4x_1 - 3x_2 - 2x_3 + 2x_4 = -1 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 3 & 0 & -2 & 1 & -2 \\ 2 & 3 & -2 & 0 & -3 \\ 1 & -3 & 0 & 1 & 1 \\ 4 & -3 & -2 & 2 & -1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 1 \\ 2 & 3 & -2 & 0 & -3 \\ 3 & 0 & -2 & 1 & -2 \\ 4 & -3 & -2 & 2 & -1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 1 \\ 0 & 9 & -2 & -2 & -5 \\ 0 & 9 & -2 & -2 & -5 \\ 0 & 9 & -2 & -2 & -5 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 1 \\ 0 & 9 & -2 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_2 = \frac{1}{9}\left[2x_3 + 2x_4 - 5\right] = \frac{1}{9}\left[2t + 2s - 5\right]$$

$$x_3 = t,\ x_4 = s$$

$$\Rightarrow x_1 = 3x_2 - x_4 + 1 = \frac{1}{3}\left[2t + 2s - 5\right] - s + 1 = \frac{1}{3}\left[2t - s - 2\right]$$

Sol: $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2t - s - 2) \\ \frac{1}{9}(2t + 2s - 5) \\ t \\ s \end{bmatrix} = \frac{1}{9}t \begin{bmatrix} 6 \\ 2 \\ 9 \\ 0 \end{bmatrix} + \frac{1}{9}s \begin{bmatrix} -3 \\ 2 \\ 0 \\ 9 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -5/9 \\ 0 \\ 0 \end{bmatrix}$$

b) $$\begin{cases} 3x - 2z + w = 0 \\ 2x + 3y + w = 3 \\ x - 3y - 2z = -1 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 3 & 0 & -2 & 1 & 0 \\ 2 & 3 & 0 & 1 & 3 \\ 1 & -3 & -2 & 0 & -1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & -1 \\ 2 & 3 & 0 & 1 & 3 \\ 3 & 0 & -2 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & -1 \\ 0 & 9 & 4 & 1 & 5 \\ 0 & 9 & 4 & 1 & 3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -3 & -2 & 0 & -1 \\ 0 & 9 & 4 & 1 & 5 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$\Rightarrow$ No solution. (Inconsistent)

c) $\begin{cases} x_1 - 2x_2 = 3 \\ x_1 + 3x_4 = 0 \\ 2x_2 - x_3 = 1 \\ 3x_1 - 2x_3 = -1 \end{cases} \Rightarrow -3 \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 1 & 0 & 0 & 3 & 0 \\ 0 & 2 & -1 & 0 & 1 \\ 3 & 0 & -2 & 0 & -1 \end{array}\right] \underset{-3}{=} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 3 & -3 \\ 0 & 2 & -1 & 0 & 1 \\ 0 & 6 & -2 & 0 & -10 \end{array}\right]$

$= -2 \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 3 & -3 \\ 0 & 0 & -1 & -3 & 4 \\ 0 & 0 & -2 & -9 & -1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 3 & -3 \\ 0 & 0 & -1 & -3 & 4 \\ 0 & 0 & 0 & -3 & -9 \end{array}\right] \rightarrow$

$-3x_4 = -9 \Rightarrow x_4 = 3$

$-x_3 - 3(3) = 4$

$x_3 = -13.$

$2x_2 + 3(3) = -3 \Rightarrow 2x_2 = -12 \Rightarrow x_2 = -6.$

$x_1 - 2(-6) = 3 \Rightarrow x_1 = -9.$

Sol: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9 \\ -6 \\ -13 \\ 3 \end{bmatrix}$

2. Determine condition of constants a, b and c so that the following systems is consistent. (5 pts)

$\begin{cases} x - 3y + 3z = a \\ x - y + z = b \\ x + 3y - 3z = c \end{cases} \Rightarrow -1 \left[\begin{array}{ccc|c} 1 & -3 & 3 & a \\ 1 & -1 & 1 & b \\ 1 & 3 & -3 & c \end{array}\right] =_{-3} \left[\begin{array}{ccc|c} 1 & -3 & 3 & a \\ 0 & 2 & -2 & b-a \\ 0 & 6 & -6 & c-a \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & -3 & 3 & a \\ 0 & -2 & -2 & b-a \\ 0 & 0 & 0 & -3b+3a+c-a \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & -3 & 3 & a \\ 0 & 2 & -2 & b-a \\ 0 & 0 & 0 & 2a-3b+c \end{array}\right]$

For the system to be consistent $\Rightarrow$

$\boxed{2a - 3b + c = 0}$

3. Determine the LU factorization of the matrix: (10 pts)

$$-2\quad A=\begin{bmatrix}3 & -1 & 2\\ 6 & -1 & 1\\ -3 & 5 & 2\end{bmatrix}$$

let $E_1=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$

$$\Rightarrow E_1 A=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1 & 2\\ 6 & -1 & 1\\ -3 & 5 & 2\end{bmatrix}=\begin{bmatrix}3 & -1 & 2\\ 0 & 1 & -3\\ 0 & 4 & 4\end{bmatrix}$$

let $E_2=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -4 & 1\end{bmatrix}$

$$\Rightarrow E_2 E_1 A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -4 & 1\end{bmatrix}\begin{bmatrix}3 & -1 & 2\\ 0 & 1 & -3\\ 0 & 4 & 4\end{bmatrix}=\begin{bmatrix}3 & -1 & 2\\ 0 & 1 & -3\\ 0 & 0 & 16\end{bmatrix}$$

$\Rightarrow$ Let $U=\begin{bmatrix}3 & -1 & 2\\ 0 & 1 & -3\\ 0 & 0 & 16\end{bmatrix}$

$\Rightarrow E_2 E_1 A = U \Rightarrow A = E_1^{-1}E_2^{-1}U.$

Where $L=E_1^{-1}E_2^{-1}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 4 & 1\end{bmatrix}$

$$=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 4 & 1\end{bmatrix}$$

Then $A=LU=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -1 & 4 & 1\end{bmatrix}\begin{bmatrix}3 & -1 & 2\\ 0 & 1 & -3\\ 0 & 0 & 16\end{bmatrix}$

4. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & x & 5 \\ -2 & -4 & 4 \end{bmatrix}$ (10 pts)

a) Find $\det(A)$
b) For what value of x will A have an inverse?
c) For those x such that A^{-1} exists, find $\det(A^{-1})$

Sol: a) $\det(A) = \begin{vmatrix} x & 5 \\ -4 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ x & 5 \end{vmatrix}$

$= 4x + 20 - 2(15 + x)$

$= 4x + 20 - 30 - 2x$

$= 2x + 10$

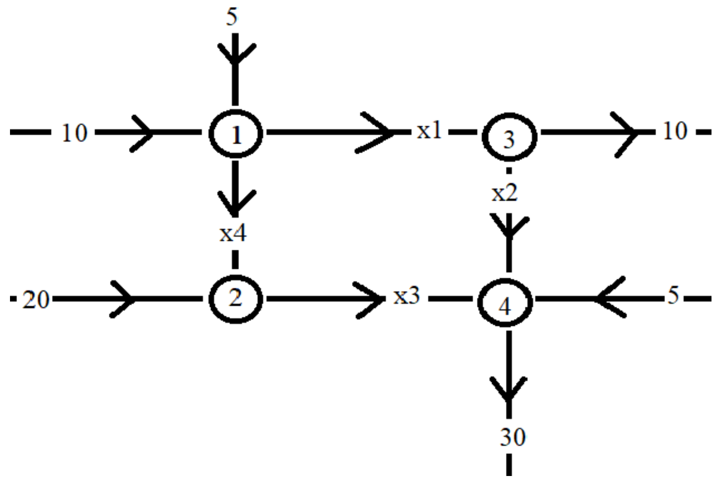
b) For A to have an inverse $\Rightarrow \det(A) \neq 0$

$\Rightarrow 2x + 10 \neq 0$

$\boxed{x \neq -5}$

c) $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{2x+10}$.

5. Using network analysis to describe the possible flows of water thru pipes as following: (10 pts)



① $10+5 = x_1 + x_4$

② $x_4 + 20 = x_3$

③ $x_1 = x_2 + 10$

④ $x_2 + x_3 + 5 = 30.$

$$\Rightarrow \begin{cases} x_1 + x_4 = 15 \\ x_3 - x_4 = 20 \\ x_1 - x_2 = 10 \\ x_2 + x_3 = 25 \end{cases} \Rightarrow \quad -1 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 0 & 1 & -1 & 20 \\ 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 25 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & -1 & 0 & -1 & -5 \\ 0 & 1 & 1 & 0 & 25 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & -1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & 20 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & -1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_3 - x_4 &= 20 \\ x_3 &= x_4 + 20 \\ x_3 &= 20 + t \\ x_4 &= t. \end{aligned}$$

$$\left.\begin{aligned} -x_2 - t = -5 &\Rightarrow x_2 = 5 - t. \\ x_1 + t = 15 &\Rightarrow x_1 = 15 - t \end{aligned}\right\} \text{sol}^{n}: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 15 - t \\ 5 - t \\ 20 + t \\ t \end{bmatrix}$$

$\Rightarrow$

6. Let $A=\begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix}_{2\times 3}$; $B=\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$; *and* $C=\begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -2 & 3 \end{bmatrix}$. Perform the following if possible. (10pts)

$a)\ AB \quad b)\ BA \quad c)\ 3A^T+C \quad d)\ B^2$

a) $A_{2\times 3} \cdot B_{2\times 2} \Rightarrow$ Impossible.

b) $B_{2\times 2}\, A_{2\times 3} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -11 & -11 & -6 \\ 10 & -6 & 4 \end{bmatrix}$

c) $3A^T + C = 3\begin{bmatrix} -2 & 3 \\ 1 & 4 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ 3 & 10 \\ -2 & 9 \end{bmatrix}$

d) $B^2 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -9 \\ -6 & 10 \end{bmatrix}$

7. a) Let A be a non-singular matrix. Prove inverse matrix is unique. (5 pts)

Suppose A has 2 inverse matrices, say B & C.

By definition of inverse matrix, we have:

$AB = BA = I$ and $AC = CA = I$.

So, start with $B = BI = B(AC)$

$\Rightarrow B = (BA)C = IC = C$.

$\Rightarrow B = C \Rightarrow$ Inverse matrix of A must be unique.

b) Let A and B are non-singular matrices. Prove that $(AB)^{-1} = B^{-1}A^{-1}$ (5 pts)

Let $C = AB$ Then $C^{-1} = (AB)^{-1}$.

claim: $B^{-1}A^{-1}$ is an inverse matrix of AB,

because $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1} I B$

$= B^{-1}B = I$.

So, we have: $(B^{-1}A^{-1})(AB) = I$.

Now consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A \; I \cdot A^{-1}$

$= A \cdot A^{-1} = I$.

$\Rightarrow (AB)(B^{-1}A^{-1}) = I$

$\Rightarrow$ By definition $\Rightarrow B^{-1}A^{-1}$ is an inverse of (AB).

Since inverse is unique $\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

8. Determine the inverse matrix of the following: (10 pts)

a) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 11 \\ 4 & -3 & 10 \end{bmatrix} \Rightarrow [A \mid I] = \begin{matrix} \\ -2 \\ -4 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 11 & 0 & 1 & 0 \\ 4 & -3 & 10 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 7 & -2 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \end{array}\right]$

$$= \begin{matrix} \\ \\ -3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 3 & 7 & -2 & 1 & 0 \end{array}\right] = \begin{matrix} \\ \\ -2 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -19 & -2 & 6 \\ 0 & 1 & 0 & -24 & -2 & 7 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -43 & -4 & 13 \\ 0 & 1 & 0 & -24 & -2 & 7 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -43 & -4 & 13 \\ -24 & -2 & 7 \\ 10 & 1 & -3 \end{bmatrix}$$

b) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 3 & 4 \end{bmatrix} \Rightarrow [A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$

$$\underset{-2}{=} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 6 & -2 & 0 & -3 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ \multicolumn{3}{c|}{\underbrace{0 \quad\quad 0 \quad\quad 0}} & -2 & 1 & 1 \end{array}\right]$$

A is a singular matrix.

9. Solve for x: $\det\begin{bmatrix} x+3 & 1 \\ -4 & x-1 \end{bmatrix} = \det\begin{bmatrix} 2x-1 & 1 \\ 2 & x+3 \end{bmatrix}$ (10pts)

$$(x+3)(x-1)+4 = (2x-1)(x+3)-2.$$

$$x^2+2x+1 = 2x^2+5x-5.$$

$$\Rightarrow x^2+3x-6=0 \Rightarrow x = \frac{-3 \pm \sqrt{9+24}}{2}.$$

$$\boxed{x = \frac{-3 \pm \sqrt{33}}{2}}$$

10. Use Cramer's Rule to solve the system: $\begin{cases} 2x-y=5 \\ x+3y=-1 \end{cases}$ (10 pts)

Let $A = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \Rightarrow \det(A) = 6+1 = 7$

$A_x = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} \Rightarrow \det(A_x) = 15-1 = 14$

$A_y = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} \Rightarrow \det(A_y) = -2-5 = -7$

$$\left.\begin{aligned} x &= \frac{\det(A_x)}{\det(A)} = \frac{14}{7} = 2. \\ y &= \frac{\det(A_y)}{\det(A)} = \frac{-7}{7} = -1 \end{aligned}\right\} \text{solution } (2,-1)$$