Exam #2 ***Math 260*** ***Name:*** KEY

10/20/10

1. ***You have 4 hours to finish this exam. Late submit exam will not be accepted.***
2. ***It's courtesy that you are the one who take this exam, please do not seek outside help of any kind.***
3. ***Scan your exam as one pdf file and submit it thru Canvas.***
4. ***Put your Full Name clearly on the first page.***

Show all your work clearly. No Work, No Credit.

1. Prove or disprove if the following set is a subspace of a vector space V. Find a basis of S if it's a vector space. (20 pts)

a) $S = \{ f(x) \in P_3 \mid f(1) + f'(1) = 0 \}$ (5 pts)

$f_1(x) \in S \Rightarrow f_1(1) + f_1'(1) = 0$

$f_2(x) \in S \Rightarrow f_2(1) + f_2'(1) = 0$

$f_1(1) + f_2(1) + f_1'(1) + f_2'(1) \stackrel{?}{=} 0$

$\Rightarrow \underbrace{f_1(1) + f_2(1)}_{0} + \underbrace{f_1'(1) + f_2'(1)}_{0} = 0$

$\Rightarrow f_1(x) + f_2(x) \in S$. ①

$\alpha f_1(1) + \alpha f_1'(1) = \alpha (f_1(1) + f_1'(1)) = \alpha \cdot 0 = 0$

$\Rightarrow \alpha \cdot f_1(x) \in S$. ②

from ① & ② $\Rightarrow S$ is a subspace

$f(x) \in P_3 \Rightarrow f(x) = ax^3 + bx^2 + cx + d$

$f(1) = a + b + c + d$

$+ \; f'(1) = 3a + 2b + c$

$f(1) + f'(1) = 4a + 3b + 2c + d = 0$

$\Rightarrow d = -4a - 3b - 2c$.

$\Rightarrow f(x) = ax^3 + bx^2 + cx + d$

$= ax^3 + bx^2 + cx - 4a - 3b - 2c$

$a(x^3 - 4) + b(x^2 - 3) + c(x - 2)$

$= \text{span}\{ x^3 - 4, x^2 - 3, x - 2 \}$.

Clearly $B = \{ x^3 - 4, x^2 - 3, x - 2 \}$ is L.I.

$\Rightarrow$ basis of S is B.

---

b) $S = \left\{ \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3 \;\middle|\; \underbrace{3a + b - c = 0}_{} \right\}$ (5 pts)

$c = 3a + b$.

Let $\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \in S \Rightarrow 3a_1 + b_1 - c_1 = 0$

$\vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \in S \Rightarrow 3a_2 + b_2 - c_2 = 0$

$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix}$ $3(a_1 + a_2) + (b_1 + b_2) - (c_1 + c_2) \stackrel{?}{=} 0$

$\underbrace{(3a_1 + b_1 - c_1)}_{0} + \underbrace{(3a_2 + b_2 - c_2)}_{0} \stackrel{?}{=} 0$

$\Rightarrow \vec{v}_1 + \vec{v}_2 \in S$. ①

$\alpha \vec{v}_1 = \begin{bmatrix} \alpha a_1 \\ \alpha b_1 \\ \alpha c_1 \end{bmatrix} \Rightarrow 3(\alpha a_1) + \alpha b_1 - \alpha c_1 \stackrel{?}{=} 0$

$\alpha (\underbrace{3a_1 + b_1 - c_1}_{0}) \stackrel{?}{=} 0$

$\alpha \cdot 0 = 0$

$\Rightarrow \alpha \vec{v}_1 \in S$. ②

$\Rightarrow S$ is a subspace of $\mathbb{R}^3$.

$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in S \Rightarrow \vec{v} = \begin{bmatrix} a \\ b \\ 3a+b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Clearly $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is L.I.

$\Rightarrow$ Basis of S is B.

c) $S = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(R) \middle| A^T = -A \right\}$ (5 pts)

let $A_1, A_2 \in S \Rightarrow A_1^T = -A_1$ & $A_2^T = -A_2$

$(A_1 + A_2)^T = A_1^T + A_2^T = -A_1 - A_2 = -(A_1 + A_2)$

$\Rightarrow A_1 + A_2 \in S$. ①

$(\alpha A_1)^T = \alpha \cdot A_1^T = \alpha(-A_1^T) = -\alpha A_1$

$\Rightarrow \alpha A_1 \in S$ ②

$\Rightarrow$ from ① & ② $\Rightarrow$ S is a subspace of $M_{2\times 2}(\mathbb{R})$

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^T = -A \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

$\Rightarrow a = -a \Rightarrow 2a = 0 \Rightarrow a = 0$

$\left. \begin{array}{l} c = -b \\ b = -c \end{array} \right\} \Rightarrow c = -b.$

$d = -d \Rightarrow 2d = 0 \Rightarrow d = 0$

$\Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$S = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$.

let $B = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \Rightarrow$ clearly B is L.I.

Basis of S is B.

---

d) $S = \{ A \in M_{n\times n} \mid \text{A is not singular} \}$ (5 pts)

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(A) = 1 \Rightarrow A \in S.$

$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \det(B) = -1 \Rightarrow B \in S.$

However $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A+B) = 0.$

$\Rightarrow A + B \notin S.$

$\Rightarrow$ S is not a subspace.

2. a) Let $B=\{2x^2-x-3,\ 3x^2+2x+4,\ 7x+17\}$.

Determine whether. $f(x)=11x^2-3x-7\in Span(B)$ (5pts)

If $f(x)\in Span(B) \Rightarrow 11x^2-3x-7=\alpha_1(2x^2-x-3)+\alpha_2(3x^2+2x+4)+\alpha_3(7x+17)$

$= (2\alpha_1+3\alpha_2)x^2+(-\alpha_1+2\alpha_2+7\alpha_3)x-3\alpha_1+4\alpha_2+17\alpha_3$

$$\begin{cases} 2\alpha_1+3\alpha_2=11 \\ -\alpha_1+2\alpha_2+7\alpha_3=-3 \\ -3\alpha_1+4\alpha_2+17\alpha_3=-7 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 11 \\ -1 & 2 & 7 & -3 \\ -3 & 4 & 17 & -7 \end{array}\right] \overset{-3}{=} \left[\begin{array}{ccc|c} -1 & 2 & 7 & -3 \\ 2 & 3 & 0 & 11 \\ -3 & 4 & 17 & -7 \end{array}\right] \underset{7}{\overset{2}{=}} \left[\begin{array}{ccc|c} -1 & 2 & 7 & -3 \\ 0 & 7 & 14 & 5 \\ 0 & -2 & -4 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -1 & 2 & 7 & -3 \\ 0 & 7 & 14 & 5 \\ 0 & 0 & 0 & 24 \end{array}\right]$$

$\Rightarrow$ No solution $\Rightarrow f(x)\notin span\{B\}$.

v_1 v_2

---

b) Let $B=\left\{\begin{bmatrix}3\\-1\\0\end{bmatrix};\begin{bmatrix}-2\\1\\3\end{bmatrix}\right\}$. Determine whether: $\vec{v}=\begin{bmatrix}11\\3\\6\end{bmatrix}\in Span(B)$ (5 pts)

$\Rightarrow$ Consider $A=\begin{bmatrix}3 & -2 & 11\\-1 & 1 & 3\\0 & 3 & 6\end{bmatrix}$

$\Rightarrow \det A = 3\begin{vmatrix}1 & 3\\3 & 6\end{vmatrix} + \begin{vmatrix}-2 & 11\\3 & 6\end{vmatrix}$

$= 3(6-9)+(-12-33)$

$= -9 - 45 = -54 \neq 0.$

Then $\{\vec{v}_1, \vec{v}_2, \vec{v}\}$ is L.I.

$\Rightarrow \vec{v}=\begin{bmatrix}11\\3\\6\end{bmatrix}\notin span\left\{\begin{bmatrix}-3\\-1\\0\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$

3. Determine whether the following is linear independent.

a) $B = \{\underbrace{5x^2 - x - 3}_{f_1(x)},\ \underbrace{-2x^2 + 3x - 1}_{f_2(x)},\ \underbrace{4x^2 + 7x - 9}_{f_3(x)}\}$ (5 pts)

$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) = \alpha_1 (5x^2 - x - 3) + \alpha_2 (-2x^2 + 3x - 1) + \alpha_3 (4x^2 + 7x - 9) = 0$

$= (5\alpha_1 - 2\alpha_2 + 4\alpha_3)x^2 + (-\alpha_1 + 3\alpha_2 + 7\alpha_3)x - 3\alpha_1 - \alpha_2 - 9\alpha_3 = 0$

$$\begin{cases} 5\alpha_1 - 2\alpha_2 + 4\alpha_3 = 0 \\ -\alpha_1 + 3\alpha_2 + 7\alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 - 9\alpha_3 = 0 \end{cases} \Rightarrow \begin{bmatrix} 5 & -2 & 4 \\ -1 & 3 & 7 \\ -3 & -1 & -9 \end{bmatrix} \overset{\cdot 5}{=} \begin{bmatrix} -1 & 3 & 7 \\ 5 & -2 & 4 \\ -3 & -1 & -9 \end{bmatrix} \overset{10}{=} \begin{array}{c} \\ \\ 13 \end{array} \begin{bmatrix} -1 & 3 & 7 \\ 0 & 13 & 39 \\ 0 & -10 & -30 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 13 & 39 \\ 0 & 0 & 0 \end{bmatrix}$$

$13\alpha_2 + 39\alpha_3 = 0 \Rightarrow B = \{f_1(x), f_2(x), f_3(x)\}$ is L.D.

b) $S = \left\{ \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ 6 \end{bmatrix} \right\}$ (5 pts)

Clearly S is Linearly dependent, because S contains more vectors than dimension of $V = \mathbb{R}^3$.

4. Find the conditions of the value(s) k so that $B=\left\{\begin{bmatrix}-1\\2\\k+1\end{bmatrix};\begin{bmatrix}0\\1\\k-3\end{bmatrix};\begin{bmatrix}1\\k\\1\end{bmatrix}\right\}$ is a linearly independent set of vectors. (5 pts)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; \vec{v}_1 \quad\;\; \vec{v}_2 \quad \vec{v}_3.$

For B to be L.I $\Rightarrow \det\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \neq 0. \Rightarrow \det\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & k \\ k+1 & k-3 & 1 \end{bmatrix} \neq 0.$

$$= -\begin{vmatrix} 1 & k \\ k-3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ k+1 & k-3 \end{vmatrix}$$

$$= -\left(1-k(k-3)\right) + \left(2(k-3)-(k+1)\right)$$

$$= -1+k^2-3k+2k-6-k-1$$

$$= k^2-2k-8 = (k-4)(k+2) \neq 0$$

$$k \neq 4, -2.$$

5. Let $B=\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\right\}$ be linearly independent set of vectors. Prove that $S=\left\{\vec{v}_1,\ \vec{v}_1+\vec{v}_2,\ \vec{v}_1+\vec{v}_2+\vec{v}_3,\ \vec{v}_1+\vec{v}_2+\vec{v}_3+\vec{v}_4\right\}$ is also a linearly independent set. (10 pts)

N.t.s. $\alpha_1\vec{v}_1 + \alpha_2(\vec{v}_1+\vec{v}_2) + \alpha_3(\vec{v}_1+\vec{v}_2+\vec{v}_3) + \alpha_4(\vec{v}_1+\vec{v}_2+\vec{v}_3+\vec{v}_4) = 0$

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\vec{v}_1 + (\alpha_2+\alpha_3+\alpha_4)\vec{v}_2 + (\alpha_3+\alpha_4)\vec{v}_3 + \alpha_4\vec{v}_4 = 0$$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is L.I,

$$\Rightarrow \left.\begin{aligned} \alpha_1+\alpha_2+\alpha_3+\alpha_4 &= 0 \\ \alpha_2+\alpha_3+\alpha_4 &= 0 \\ \alpha_3+\alpha_4 &= 0 \\ \alpha_4 &= 0 \end{aligned}\right\} \Rightarrow \text{clearly } \alpha_1=\alpha_2=\alpha_3=\alpha_4=0$$

$\Rightarrow S$ is L.I.

6. Find a minimal set of the B which is linearly independent.

a) $$B=\left\{\underbrace{\begin{bmatrix}2\\-1\\0\end{bmatrix}}_{v_1};\underbrace{\begin{bmatrix}3\\0\\1\end{bmatrix}}_{v_2},\underbrace{\begin{bmatrix}6\\0\\2\end{bmatrix}}_{v_3};\underbrace{\begin{bmatrix}3\\-2\\1\end{bmatrix}}_{v_4}\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{v_5}\right\}$$ (5 pts)

Observe that $\vec{v_5}=\vec{v_2}-\vec{v_1}=\begin{bmatrix}3\\0\\1\end{bmatrix}-\begin{bmatrix}2\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow$ drop $\vec{v_5}$.

$\vec{v_3}=2\vec{v_2}=2\begin{bmatrix}3\\0\\1\end{bmatrix}=\begin{bmatrix}6\\0\\2\end{bmatrix} \Rightarrow$ drop $\vec{v_3}$.

for $A=\begin{bmatrix}| & | & |\\ v_1 & v_2 & v_4\\ | & | & |\end{bmatrix} \Rightarrow \det(A)=\det\begin{bmatrix}2 & 3 & 3\\ -1 & 0 & -2\\ 0 & 1 & 1\end{bmatrix}$

$\Rightarrow \det A = 2\begin{vmatrix}0 & -2\\ 1 & 1\end{vmatrix}+\begin{vmatrix}3 & 3\\ 1 & 1\end{vmatrix}=2(0+2)+(3-3)=4\neq 0$

$\Rightarrow$ minimal set $\{\vec{v_1},\vec{v_2},\vec{v_4}\}$

b) $$B=\{\underbrace{\cancel{2x-1}}_{f_1(x)},\underbrace{x^2+x+1}_{f_2(x)};\underbrace{\cancel{2x^2+2x+2}}_{f_3(x)};\underbrace{x^2+3x}_{f_4(x)};\underbrace{x-3}_{f_5(x)}\}$$ (5pts)

clearly $2f_2(x)=2(x^2+x+1)=2x^2+2x+2=f_3(x) \Rightarrow$ drop $f_3(x)$

$f_4(x)-f_2(x)=x^2+3x-(x^2+x+1)=2x-1=f_1(x) \Rightarrow$ drop $f_1(x)$

Consider new set $\{f_2(x), f_4(x), f_5(x)\} \Rightarrow \alpha_1 \cdot f_2+\alpha_2 f_4+\alpha_3 f_5=0$

$\alpha_1(x^2+x+1)+\alpha_2(x^2+3x)+\alpha_3(x-3)=0$

$(\alpha_1+\alpha_2)x^2+(\alpha_1+3\alpha_2+\alpha_3)x+\alpha_1-3\alpha_3=0$

$\begin{cases}\alpha_1+\alpha_2=0\\ \alpha_1+3\alpha_2+\alpha_3=0\\ \alpha_1-3\alpha_2=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 & 0\\ 1 & 3 & 1\\ 1 & -3 & 0\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 1\\ 0 & -4 & 0\end{bmatrix} \Rightarrow \alpha_1=\alpha_2=\alpha_3=0$

$\Rightarrow$ minimal set $\{f_2(x), f_4(x), f_5(x)\}$

7. Find a basis of a subspace S of a vector space V, then extend that basis to a basis of the vector space V. (20 pts)

a) $B = \left\{ f(x) = ax^2 + bx + c \in P_2 \,\middle|\, \int_1^2 f(x)\,dx = 0 \right\}$

$f(x) = ax^2 + bx + c \in B \Rightarrow \int_1^2 (ax^2 + bx + c)\,dx = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx \Big|_1^2 = 0$

$= \frac{8}{3}a + 2b + 2c - \frac{1}{3}a - \frac{1}{2}b - c = 0.$

$\frac{7}{3}a + \frac{3}{2}b + c = 0 \Rightarrow c = -\frac{7}{3}a - \frac{3}{2}b.$

$f(x) = ax^2 + bx + c = ax^2 + bx - \frac{7}{3}a - \frac{3}{2}b$

$= a\left(x^2 - \frac{7}{3}\right) + b\left(x - \frac{3}{2}\right) = \text{Span}\left\{x^2 - \frac{7}{3}, x - \frac{3}{2}\right\}$

let $A = \left\{x^2 - \frac{7}{3}, x - \frac{3}{2}\right\}$.

Clearly, A is L.I.

$\Rightarrow$ Basis of B is $A = \left\{x^2 - \frac{7}{3}, x - \frac{3}{2}\right\}$ to extend to a basis of P_2

we need one more function $f(x)$ such that $\int_1^2 f(x)d \neq 0$, let $f(x) = 1$.

Since $\int_1^2 f(x)dx = \int_1^2 1\,dx = 2 - 1 = 1 \neq 0 \Rightarrow$ Extended basis: $\left\{x^2 - \frac{7}{3}, x - \frac{3}{2}, 1\right\}$

b) $B = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(R) \,\middle|\, a + 3b = 2c - d \right\}$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in B \Rightarrow a + 3b = 2c - d$

$\Rightarrow a = 2c - d - 3b.$

$\Rightarrow A = \begin{bmatrix} 2c - d - 3b & b \\ c & d \end{bmatrix} = c\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ c & 1 \end{bmatrix}.$

$= \text{Span}\left\{ \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_3} \right\}$

Basis : $= \{A_1, A_2, A_3\} \Rightarrow$ Extend the basis $\Rightarrow$ find $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + 3b \neq 2c - d \Rightarrow a \neq -3b + 2c - d \Rightarrow$ Choose $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$\Rightarrow 1 \neq 3(0) + 2(0) - 0 = 0. \Rightarrow$ Extended basis: $\Rightarrow \left\{A_1, A_2, A_3, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

c) $S = \{f(x) \in P_2 | f(1) + 2f'(1) = 0\}$

Let $f(x) = ax^2 + bx + c \in S$.

$f(1) = a + b + c$

$2f'(1) = 2(2a+b)$

$f(1) + 2f'(1) = 5a + 3b + c = 0.$

$\Rightarrow c = -5a - 3b$

$\Rightarrow f(x) = ax^2 + bx + c$

$= ax^2 + bx - 5a - 3b.$

$= a(x^2 - 5) + b(x - 3)$

$= \text{span}\{x^2 - 5, x - 3\}$

$\Rightarrow$ Basis of $S = \{x^2 - 5, x - 3\}$ $\Rightarrow$ Extended basis of S $\Rightarrow$ find a function $f(x)$ such that $f(1) + 2f'(1) \neq 0$ $\Rightarrow$ choose $f(x) = 1.$ Extended basis from S.

$\Rightarrow f(1) + 2f'(1) = 1 + 2(0) = 1 \neq 0, \Rightarrow$

$B = \{x^2 - 5, x - 3, 1\}$

d) $S = \left\{ \vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in R^4 \middle| 3a - d = 2c - 3b \right\} \Rightarrow \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in R^4 \middle| d = 3a - 2c + 3b, \right\}$

$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 3a - 2c + 3b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}.$

$\Rightarrow \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}}_{= v_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}}_{= v_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}}_{= v_3.} \right\}. \Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is L.I.

$\Rightarrow$ Basis of $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$ $\Rightarrow$ Extended basis $\Rightarrow$ choose

$v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $d \neq 3a - 2c + 3b$

$\Rightarrow$ Choose $d = 1$

$a = c = b = 0$ $\rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

$\Rightarrow$ Extended basis: $\left\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

8. Determine the rank, the nullity, a basis of Rowspace(A), Colspace(A) and Nullspace(A) of the following matrices. (15 pts)

a) $A=\begin{bmatrix}2&-1&3&1\\3&2&-1&5\\3&-5&10&-2\\5&-6&13&-1\end{bmatrix} = \begin{bmatrix}2&-1&3&1\\0&-7&11&-7\\0&7&-11&7\\0&7&-11&7\end{bmatrix}$

$= \begin{bmatrix}2&-1&3&1\\0&-7&11&-7\\0&0&0&0\\0&0&0&0\end{bmatrix} \Rightarrow \text{Rowspace} = \text{Span}\left\{\begin{bmatrix}2\\-1\\3\\1\end{bmatrix}, \begin{bmatrix}0\\-7\\11\\-7\end{bmatrix}\right\} \Rightarrow \text{Rank}(A) = 2.$

$\text{colspace} = \text{span}\left\{\begin{bmatrix}2\\3\\3\\5\end{bmatrix}, \begin{bmatrix}-1\\2\\-5\\-6\end{bmatrix}\right\}.$

$7y+11z-w=0 \Rightarrow w = 7t+11s \quad \begin{cases} y=t \\ z=s \end{cases}$

$\left.\begin{aligned} 2x - t + 3s + 7t + 11s &= 0 \\ 2x &= -6t - 14s \\ x &= -3t - 7s. \end{aligned}\right\}$ $\vec{v} \in \text{Nullspace}(A) \Rightarrow \vec{v} = \begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}-3t-7s\\t\\s\\7t+11s\end{bmatrix} = t\begin{bmatrix}-3\\1\\0\\7\end{bmatrix} + s\begin{bmatrix}-7\\0\\1\\11\end{bmatrix}$

$= \text{Span}\left\{\begin{bmatrix}-3\\1\\0\\7\end{bmatrix}, \begin{bmatrix}-7\\0\\1\\11\end{bmatrix}\right\}.$

Nullity $= 2$.

b) $A=\begin{bmatrix}-1&0&-3\\2&-3&0\\3&1&11\end{bmatrix} = \begin{bmatrix}-1&0&-3\\0&-3&-6\\0&1&2\end{bmatrix} = \begin{bmatrix}-1&0&-3\\0&1&2\\0&1&2\end{bmatrix}$

$= \begin{bmatrix}-1&0&-3\\0&1&2\\0&0&0\end{bmatrix}$

$\text{rowspace}(A) = \text{Span}\left\{\begin{bmatrix}-1\\0\\-3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$

$\text{Colspace}(A) = \text{Span}\left\{\begin{bmatrix}-1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\-3\\1\end{bmatrix}\right\}$

$\text{Rank}(A) = 2.$

$$\text{Nullspace}(A) \Rightarrow \begin{bmatrix} -1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad y + 2z = 0 \Rightarrow y = -2z = -2t \quad \{z = t\}$$

$$\& \quad -x - 3z = 0 \Rightarrow x = -3t.$$

$$\Rightarrow \text{Nullspace}(A) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{Nullspace}(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \text{Nullity}(A) = 1.$$