***Exam #3*** ***Math 260*** ***Name:*** KEY

***11/17/20***

1. ***You have 3 hours to finish this exam. Submitted by email will not be accepted.***
2. ***It's courtesy that you are the one who take this exam, do not seek outside help of any kind, please clear your desk, no text book, no notes, no online searching of any kind while you are taking the exam.***
3. ***Scan your exam as one pdf file and submit it thru Canvas.***
4. ***Put your Full Name clearly on the first page.***
5. ***Exam is due by 6:45pm today. Extended time 30 minutes till 7:15pm for late penalty of 25% of your score.***
6. ***Show your work clearly. No Work, No Credit.***

1. Let $B=\{3x^2-x+2,\ 2x^2-1,\ x-3\}$ be a basis of $P_2$. Determine $\left[f(x)\right]_B$ *where* $f(x)=2x^2-x+3$ (5 pts)

$$[f(x)]_B = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} \text{ where } f(x) = \alpha_1(3x^2-x+2)+\alpha_2(2x^2-1)+\alpha_3(x-3)$$
$$= (3\alpha_1+2\alpha_2)x^2+(-\alpha_1+\alpha_3)x+2\alpha_1-\alpha_2-3\alpha_3$$

$$\Rightarrow \begin{cases} 3\alpha_1+2\alpha_2=2 \\ -\alpha_1+\alpha_3=-1 \\ 2\alpha_1-\alpha_2-3\alpha_3=3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 3 & 2 & 0 & 2 \\ 2 & -1 & -3 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 2 & 3 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left.\begin{array}{l} \alpha_3=1 \\ -\alpha_2-1=1 \Rightarrow \alpha_2=-2 \\ -\alpha_1+1=-1 \Rightarrow \alpha_1=2 \end{array}\right\} \Rightarrow [f(x)]_B = \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

2. Given . $B=\{\overbrace{2+x^2}^{f_1},\ \overbrace{-1-6x+8x^2}^{f_2},\ \overbrace{-7-3x-9x^2}^{f_3}\}$ *and* $C=\{1+x,\ -x+x^2,\ 1+2x^2\}$ Find the change – of – basis matrix $P_{B\to C}$ (5 pts)

$$P_{B\to C} = \left[\,[f_1]_C \ [f_2]_C \ [f_3]_C\,\right] \text{ where } [f_i]_C = \alpha_1(1+x)+\alpha_2(-x+x^2)+\alpha_3(1+2x^2) = (\alpha_1+\alpha_3)+(\alpha_1-\alpha_2)x+(\alpha_2+2\alpha_3)x^2 \Rightarrow \begin{cases}\alpha_1+\alpha_3 = 2,-1,-7 \\ \alpha_1-\alpha_2 = 0,-6,-3 \\ \alpha_2+2\alpha_3 = 1,8,-9.\end{cases}$$

$$-1\left[\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 2 & -1 & -7 \\ 1 & -1 & 0 & 0 & -6 & -3 \\ 0 & 1 & 2 & 1 & 8 & -9 \end{array}\right] = \left[\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 2 & -1 & -7 \\ 0 & -1 & -1 & -2 & -5 & 4 \\ 0 & 1 & 2 & 1 & 8 & -9 \end{array}\right] = \left[\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 2 & -1 & -7 \\ 0 & -1 & -1 & -2 & -5 & 4 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right]$$

$$\left.\begin{array}{rcl|c|c} \alpha_3 & = & -1 & 3 & -5 \\ -\alpha_2+1 & = & -2 & -5 & 4 \Rightarrow \\ \alpha_2 & = & 3 & 6 & -3 \\ \alpha_1-1 & = & 2 & -1 & -7 \\ \alpha_1 & = & 3 & 0 & -6 \end{array}\right\} \Rightarrow P_{B\to C} = \begin{bmatrix} 3 & 0 & -6 \\ 3 & 6 & -3 \\ -1 & 3 & -5 \end{bmatrix}$$

3. Let $P_2$, define a map $\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$ (15pts)

a) Prove that $\{P_2, \langle p(x), q(x)\rangle\}$ is an inner product vector space.

b) Determine the angle between $p(x) = 3x^2 + 1$ and $q(x) = 2x - 3$

c) Determine $proj_{p(x)} q(x)$

1) $\langle p(x), p(x)\rangle = (p(0))^2 + (p(1))^2 + (p(2))^2 \geq 0$

2) $\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) = q(0)p(0) + q(1)p(1) + q(2)p(2) = \langle q(x), p(x)\rangle$

3) $\langle \alpha p(x), q(x)\rangle = \alpha p(0)q(0) + \alpha p(1)q(1) + \alpha p(2)q(2) = \alpha\,(p(0)q(0) + p(1)q(1) + p(2)q(2)) = \alpha \langle p(x), q(x)\rangle$

4) $\langle p(x), q(x)+r(x)\rangle = p(0)\cdot(q(0)+r(0)) + p(1)(q(1)+r(1)) + p(2)(q(2)+r(2))$

$= [p(0)q(0) + p(1)q(1) + p(2)q(2)] + [p(0)r(0) + p(1)r(1) + p(2)r(2)] = \langle p(x), q(x)\rangle + \langle p(x), r(x)\rangle$

$\Rightarrow \{P_2, \langle p, q\rangle\}$ is an IPVS.

b) Angle $\theta$ between $p(x)$ & $q(x)$: $\cos\theta = \dfrac{\langle p(x), q(x)\rangle}{\|p(x)\|\cdot\|q(x)\|}$.

$\langle p(x), q(x)\rangle = (1)(-3) + (4)(-1) + (13)(1) = 6$

$\|p(x)\| = \sqrt{(1)^2 + (4)^2 + (13)^2} = \sqrt{186}$

$\|q(x)\| = \sqrt{(-3)^2 + (-1)^2 + (1)^2} = \sqrt{11}$

$\Rightarrow \cos\theta = \dfrac{6}{\sqrt{186}\cdot\sqrt{11}} = \dfrac{6}{\sqrt{2046}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{6}{\sqrt{2046}}\right) = 1.44\text{ rad} \approx 82.5^\circ.$

c) $\text{Proj}_{p(x)} q(x) = \dfrac{\langle p(x), q(x)\rangle}{\|p(x)\|^2}\cdot p(x)$

$\langle p(x), q(x)\rangle = 6$

$\|p(x)\|^2 = (\sqrt{186})^2 = 186.$

$$\text{Proj}_{p(x)} q(x) = \frac{6}{186}(3x^2+1) = \frac{3}{31}x^2 + \frac{1}{31}.$$

4. Let $\{V,\langle,\rangle\}$ be an inner – product vector space and let $\vec{v} \in V$. Let $S^{\perp}$ be the set of all vectors in V that are orthogonal to $\vec{v}$, i.e. $S^{\perp} = \left\{ \vec{u} \in V \middle| \left\langle \vec{v}, \vec{u} \right\rangle = 0 \right\}$. Prove that $S^{\perp}$ is a subspace of V. (5 pts)

Prf: Let $\vec{u}_1, \vec{u}_2 \in S^{\perp} \xRightarrow{\text{N.t.S.}} \vec{u}_1 + \vec{u}_2 \in S^{\perp}$. i.e. N.t.S. $\langle \vec{v}, \vec{u}_1 + \vec{u}_2 \rangle = 0$.

Since $\vec{u}_1 \in S^{\perp} \Rightarrow \langle \vec{v}, \vec{u}_1 \rangle = 0$

& $\vec{u}_2 \in S^{\perp} \Rightarrow \langle \vec{v}, \vec{u}_2 \rangle = 0$

let $\vec{u} = \vec{u}_1 + \vec{u}_2 \Rightarrow \langle \vec{v}, \vec{u}_1 + \vec{u}_2 \rangle = \langle \vec{v}, \vec{u}_1 \rangle + \langle \vec{v}, \vec{u}_2 \rangle$

$\Rightarrow \langle \vec{v}, \vec{u}_1 + \vec{u}_2 \rangle = 0 + 0 = 0$

$\Rightarrow \vec{u}_1 + \vec{u}_2 \in S^{\perp}$ (1)

let $\alpha \in \mathbb{R} \xRightarrow{\text{N.t.S.}} \alpha \vec{u}_1 \in S^{\perp} \Rightarrow$ i.e. $\langle \vec{v}, \alpha \vec{u}_1 \rangle \overset{?}{=} 0$.

We have: $\langle \vec{v}, \alpha \vec{u}_1 \rangle = \langle \alpha \vec{u}_1, \vec{v} \rangle =$

$= \alpha \langle \vec{u}_1, \vec{v} \rangle = \alpha \langle \vec{v}, \vec{u}_1 \rangle = \alpha \cdot 0 = 0$

$\Rightarrow \alpha \vec{u}_1 \in S^{\perp}$ (2)

from (1) & (2) $\Rightarrow S^{\perp}$ is a subspace of V.

5. Let $T: V \mapsto W$ be an injective linear transformation from vector space V to a vector space W. Prove that if $B = \left\{ \vec{v_1}, \vec{v_2}, \vec{v_3}, \ldots, \vec{v_n} \right\}$ is a set of linearly independent vectors in V then $C = \left\{ T\left(\vec{v_1}\right), T\left(\vec{v_2}\right), T\left(\vec{v_3}\right), \ldots, T\left(\vec{v_n}\right) \right\}$ is a set of linearly independent vectors in W. (5 pts)

lecture notes.

6. Let $T: V \mapsto W$ be a linear transformation from vector space V to a vector space W. Prove the following statements: (10 pts)

a) $Ker(T)$ is a subspace of V.

Lecture notes

b) $\mathrm{Im}(T)$ is a subspace of W.

Lecture notes.

7. Find a linear transformation $T: R^2 \mapsto R^3$ with $T\begin{bmatrix}-3\\2\end{bmatrix}=\begin{bmatrix}-1\\0\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right)=\begin{bmatrix}1\\-1\\3\end{bmatrix}$. (10 pts)

Want to define $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \vec{v} = \begin{bmatrix}?\end{bmatrix} \in \mathbb{R}^3$.

Given any vector $\vec{v} = \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \alpha_1 \begin{bmatrix}-3\\2\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-1\end{bmatrix}$.

$$\Rightarrow \begin{matrix}2\\3\end{matrix}\left[\begin{array}{cc|c}-3 & 2 & x\\ 2 & -1 & y\end{array}\right] = \left[\begin{array}{cc|c}-3 & 2 & x\\ 0 & 1 & 2x+3y\end{array}\right]$$

$\alpha_2 = 2x+3y$

$-3\alpha_1 + 4x+6y = x$

$-3\alpha_1 = -3x-6y \Rightarrow \alpha_1 = x+2y$

So for any vector $\vec{v} = \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$

$\Rightarrow \vec{v} = \alpha_1 \begin{bmatrix}-3\\2\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-1\end{bmatrix} = (x+2y)\begin{bmatrix}-3\\2\end{bmatrix} + (2x+3y)\begin{bmatrix}2\\-1\end{bmatrix}$.

$\Rightarrow T(\vec{v}) = T\begin{pmatrix}x\\y\end{pmatrix} = (x+2y)\,T\begin{pmatrix}-3\\2\end{pmatrix} + (2x+3y)\,T\begin{pmatrix}2\\-1\end{pmatrix} = (x+2y)\begin{bmatrix}-1\\0\\2\end{bmatrix} + (2x+3y)\begin{bmatrix}1\\-1\\3\end{bmatrix}$

$= \begin{bmatrix}-x-2y+2x+3y\\ 0-2x-3y\\ 2x+4y+6x+9y\end{bmatrix} = \begin{bmatrix}x+y\\ -2x-3y\\ 8x+13y\end{bmatrix}$

8. Determine the Ker(T) and Im(T) of the following linear transformations, and then indicate if T is injective, surjective or an isomorphism. (20 pts)

a) $T: M_2(R) \mapsto R^3$ by $T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \begin{bmatrix}a-3d\\ a+b\\ d+4c\end{bmatrix}$

$\ker(T) = \left\{ A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in M_2(\mathbb{R}) \;\middle|\; T(A) = \begin{bmatrix}0\\0\\0\end{bmatrix} \in \mathbb{R}^3 \right\} \Rightarrow T(A) = \begin{bmatrix}a-3d\\ a+b\\ d+4c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{cases}a-3d=0\\ a+b=0\\ d+4c=0\end{cases}$

$\left.\begin{array}{l} d=4c\\ a=-3d=-3(4c)=12c\\ b=-a=-12c\end{array}\right\} \Rightarrow A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \ker(T) \Rightarrow A = \begin{bmatrix}12c & -12c\\ c & 4c\end{bmatrix} = c\begin{bmatrix}12 & -12\\ 1 & 4\end{bmatrix}$.

$\Rightarrow \ker(T) = \text{Span}\left(\begin{bmatrix}12 & -12\\ 1 & 4\end{bmatrix}\right\} \Rightarrow$ Nullity $(T) = 1$. (Not Injective).

$\text{Im}(T) = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \exists A \in M_2(\mathbb{R}) \text{ such that } T(A) = \begin{bmatrix}x\\y\\z\end{bmatrix} \right\}$.

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}a-3d\\ a+b\\ d+4c\end{bmatrix} = a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\4\end{bmatrix} + d\begin{bmatrix}-3\\0\\1\end{bmatrix} = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\4\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$

We have: Nullity $(T)$ + Rank$(T) = 4 \Rightarrow$ Rank$(T) = 3 \Rightarrow \text{Im}(T) = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\4\end{bmatrix}\right\}$

b/c $A = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 4\end{bmatrix} \Rightarrow \det(A) = 4 \Rightarrow$ T is surjective.

b) $T: R^3 \mapsto P_2 \quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (2a+b)x^2 + (c-b)x + a + b + c$

$\ker(T) = \left\{ \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; T(\vec{v}) = 0x^2 + 0x + 0 \in \underline{P}_2 \right\}$

$T(v) = (2a+b)x^2 + (c-b)x + a+b+c$

$\Rightarrow \begin{cases} 2a+b=0 \\ c-b=0 \\ a+b+c=0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$

$\Rightarrow \ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow T$ is injective.

$\mathrm{Im}(T) = \left\{ mx^2 + nx + p \in \underline{P}_2 \;\middle|\; \exists \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = mx^2 + nx + p \right\}$

$mx^2 + nx + p = (2a+b)x^2 + (c-b)x + a+b+c$
$= a(2x^2+1) + b(x^2-x+1) + c(x+1) = \mathrm{span}\{2x^2+1, x^2-x+1, x+1\}$

$\Rightarrow \mathrm{Nullity}(T) = 0 \Rightarrow \mathrm{Rank}(T) = 3 \Rightarrow \mathrm{Im}(T) = \mathrm{span}\{2x^2+1, x^2-x+1, x+1\} = \underline{P}_2$

$\Rightarrow T$ is surjective.

$\Rightarrow T$ is an isomorphism. (bijective)

c) $T: P_2 \mapsto R^3$ by $T\left(ax^2+bx+c\right)=\begin{bmatrix}3a-b\\ a+b+c\\ 4a+c\end{bmatrix}$

$\ker(T)=\left\{ f(x)=ax^2+bx+c \in P_2 \;\middle|\; T(f(x))=\begin{bmatrix}0\\0\\0\end{bmatrix}\in R^3 \right\} \Rightarrow T(f(x))=\begin{bmatrix}3a-b\\ a+b+c\\ 4a+c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{cases}3a-b=0\\ a+b+c=0\\ 4a+c=0\end{cases}$

$\begin{bmatrix}1&1&1\\3&-1&0\\4&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\0&-4&-3\\0&-4&-3\end{bmatrix}=\begin{bmatrix}1&1&1\\0&4&3\\0&0&0\end{bmatrix} \begin{cases}\rightarrow 4b+3c=0 \Rightarrow b=-\frac{3}{4}c=-\frac{3}{4}t \quad \{c=t\}\\ a+b+c=0 \Rightarrow a=\frac{3}{4}t-t=-\frac{1}{4}t\end{cases}$

$\ker(T)=ax^2+bx+c=-\frac{1}{4}tx^2-\frac{3}{4}tx+t=t\left(-\frac{1}{4}x^2-\frac{3}{4}x+1\right)=\text{Span}\left\{-\frac{1}{4}x^2-\frac{3}{4}x+1\right\}$

$\text{Im}(T)=\left\{\begin{bmatrix}m\\n\\p\end{bmatrix}\in \mathbb{R}^3 \;\middle|\; \exists f(x)=ax^2+bx+c,\ T(f(x))=\begin{bmatrix}m\\n\\p\end{bmatrix}\right\}$

$\Rightarrow \begin{bmatrix}m\\n\\p\end{bmatrix}=\begin{bmatrix}3a-b\\a+b+c\\4a+c\end{bmatrix}=a\begin{bmatrix}3\\1\\4\end{bmatrix}+b\begin{bmatrix}-1\\1\\0\end{bmatrix}+c\begin{bmatrix}0\\1\\1\end{bmatrix}=\text{span}\left\{\begin{bmatrix}3\\1\\4\end{bmatrix},\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

$\Rightarrow \text{Rank}(T)=2 \Rightarrow \text{Im}(T)=\text{span}\left\{\begin{bmatrix}3\\1\\4\end{bmatrix},\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}. \Rightarrow T$ is not surjective.

$\text{Nullity}(T)=1 \quad + \text{rank}(T)=3, \Rightarrow$

$\Rightarrow$

d) $T: R^3 \mapsto R^2$ by $T(\vec{v})=A\vec{v}$ where $A=\begin{bmatrix}1&-3&2\\-3&9&-6\end{bmatrix}$

$\ker(T)=\left\{\vec{v}=\begin{bmatrix}x\\y\\z\end{bmatrix}\in \mathbb{R}^3 \quad T(\vec{v})=\begin{bmatrix}0\\0\end{bmatrix}\in \mathbb{R}^2\right\}$

$T(\vec{v})=A\vec{v}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \ker(T)=\text{Nullspace}(A) \Rightarrow \begin{bmatrix}1&-3&2\\-3&9&-6\end{bmatrix}=\begin{bmatrix}1&-3&2\\0&0&0\end{bmatrix}$

$\Rightarrow x-3y+2z=0 \Rightarrow x=3y-2z.$

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3y-2z\\y\\z\end{bmatrix}=y\begin{bmatrix}3\\1\\0\end{bmatrix}+z\begin{bmatrix}-2\\0\\1\end{bmatrix}=\text{span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right\} \Rightarrow$ Not injective.

$\text{Im}(T)=\text{colspace}(A)=\text{span}\left\{\begin{bmatrix}1\\-3\end{bmatrix}\right\} \Rightarrow$ Not surjective.

9. Let $T_1 : M_{2\times2}(R) \mapsto P_2$ by $T_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-c)+(b+2c)x+(3c-d)x^2$ (10 pts)

And $T_2 : P_2 \mapsto R^2$ by $T\left(ax^2+bx+c\right) = \begin{bmatrix} a+b \\ b-2c \end{bmatrix}$. Determine a formula for the composition linear transformation $T_2T_1 : M_{2\times2}(R) \mapsto R^2$. Then determine a basis for $Ker(T_2 \cdot T_1)$ and $\operatorname{Im}(T_2 \cdot T_1)$

$T_2 \cdot T_1 : M_{2\times2} \longmapsto \mathbb{R}^2 \Rightarrow T_2 \cdot T_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T_2\left(a-c+(b+2c)x+(3c-d)x^2\right) = \begin{bmatrix} 3c-d+b+2c \\ b+2c-2(a-c) \end{bmatrix}$

$\Rightarrow T_2 \cdot T_1(A) = \begin{bmatrix} b+5c-d \\ -2a+b+4c \end{bmatrix}$

$\ker(T_2 \cdot T_1) = \left\{ A \in M_{2\times2} \mid T_2 \cdot T_1(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \begin{cases} b+5c-d=0 \Rightarrow d=b+5c \\ -2a+b+4c=0 \Rightarrow a=\frac{1}{2}b+2c. \end{cases}$

$\Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T_2 \cdot T_1) = \begin{bmatrix} \frac{1}{2}b+2c & b \\ c & b+5c \end{bmatrix} = b\begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \right\}$

$\operatorname{Im}(T_2 \cdot T_1) = \left\{ \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{R}^2 \mid \exists A \in M_{2\times2};\ T(A) = \begin{bmatrix} m \\ n \end{bmatrix} \right\}$

$\Rightarrow \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} b+5c-d \\ -2a+b+4c \end{bmatrix} = a\begin{bmatrix} 0 \\ -2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 5 \\ 4 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$= \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \Rightarrow$ surjective.

10. Find the matrix representation $[T]_B^C$ (15 pts)

a) $T : P_3 \mapsto P_2$ by $T(y) = 3y'' + 2y'$ for all $y \in P_3$, where $B = \{1, x, x^2, x^3\}$ and $C = \{1, x, x^2\}$

$$[T]_B^C = \begin{bmatrix} [T(1)]_C & [T(x)]_C & [T(x^2)]_C & [T(x^3)]_C \end{bmatrix}.$$

where $T(1) = 3(1)'' + 2(1)' = 0.$

$T(x) = 3(x)'' + 2(x)' = 2.$

$T(x^2) = 3(x^2)'' + 2(x^2)' = 12 + 4x$

$T(x^3) = 3(x^3)'' + 2(x^3)' = 36x + 6x^2$

$$\Rightarrow [T]_B^C = \begin{bmatrix} 0 & 2 & 12 & 0 \\ 0 & 0 & 4 & 18 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

$[T(1)]_C = [0]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $[T(x)]_C = [2]_C = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$[T(x^2)]_C = [12 + 4x]_C = \begin{bmatrix} 12 \\ 4 \\ 0 \end{bmatrix}$, $[T(x^3)]_C = [18x + 16x^2]_C = \begin{bmatrix} 0 \\ 18 \\ 16 \end{bmatrix}$

$0x^3$

$2(3x^2)$

$3(6x)$

b) $T : M_{2\times2} \mapsto M_{2\times2}$ by $T(A) = 2A + A^T$ where

$B = C =$ standard basis of $M_{2\times2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$A_1, \quad A_2, \quad A_3, \quad A_4$

$$[T]_B^C = \begin{bmatrix} [T(A_1)]_C & [T(A_2)]_C & [T(A_3)]_C & [T(A_4)]_C \end{bmatrix}$$

$T(A_1) = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}_C = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T(A_2) = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$

$T(A_3) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$T(A_4) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

$$\Rightarrow [T]_B^C = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

c) Let $V = span\{e^x, xe^x, x^2e^{3x}\}$ define $T : V \mapsto V$ by $T(f(x)) = \frac{df}{dx}$ where $B = C = \{e^x, xe^x, x^2e^x\}$

$$[T]_B^C = \begin{bmatrix} | & | & | \\ [T(e^x)]_C & [T(xe^x)]_C & [T(x^2e^x)]_C \\ | & | & | \end{bmatrix}$$

Where $T(e^x) = \frac{d}{dx}(e^x) = e^x \Rightarrow [e^x]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$T(xe^x) = \frac{d}{dx}(xe^x) = e^x(x+1) \Rightarrow [e^x + xe^x]_C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(x^2e^x) = \frac{d}{dx}(x^2e^x) = e^x(x^2+2x) = [2xe^x + x^2e^x]_C = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

$$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$