9/15/20

Solve the following systems of linear equations by REF: (7 pts)

$$\begin{cases} x_1 - 2x_2 + 5x_4 = 1 \\ 2x_1 - x_3 - x_4 = 0 \\ 3x_1 - 2x_2 - x_3 + 4x_4 = 1 \\ 4x_1 - 4x_2 - x_3 + 9x_4 = 2 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & 5 & 1 \\ 2 & 0 & -1 & -1 & 0 \\ 3 & -2 & -1 & 4 & 1 \\ 4 & -4 & -1 & 9 & 2 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & -2 & 0 & 5 & 1 \\ 0 & 4 & -1 & -11 & -2 \\ 0 & 4 & -1 & -11 & -2 \\ 0 & 4 & -1 & -11 & -2 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 5 & 1 \\ 0 & 4 & -1 & -11 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow 4x_2 - x_3 - 11x_4 = -2 \Rightarrow x_3 = 4x_2 - 11x_4 + 2$$

let $x_2 = t$, $x_4 = s$.

$$\Rightarrow x_3 = 4t - 11s + 2.$$

$$x_1 - 2t + 5s = 1 \Rightarrow x_1 = 2t - 5s + 1$$

Sol: $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2t - 5s + 1 \\ t \\ 4t - 11s + 2 \\ s \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 0 \\ -11 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Find conditions of real numbers a, b, c such that $\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 3 & 4 & 7 & b \\ 1 & 2 & 3 & c \end{array}\right]$ has a solution. (8 pts)

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & b - 3a \\ 0 & 1 & 1 & c - a \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & b - 3a \\ 0 & 0 & 0 & -b + 3a + c - a \end{array}\right]$$

For consistent $\Rightarrow -b + 3a + c - a = 0$

$$\boxed{2a - b + c = 0}$$

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4. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$; $B = \begin{bmatrix} -2 & 3 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -3 \end{bmatrix}$. Perform the following: (15pts)

a) AB b) $A + 2C^T$ c) $B^T C$ d) A^{-1} e) $\det(C)$

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 0 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -2 & -11 \\ -5 & -14 \end{bmatrix}$$

$$A + 2C^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 & 0 \\ -3 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ -2 & 3 & 8 \\ 7 & 8 & 3 \end{bmatrix}$$

3×2 $\quad$ 2×3

$$\det(C) = \begin{vmatrix} 1 & -3 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -3 \end{vmatrix} = -3\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix}$$

$$= -3(-1 + 6) = \boxed{-15}$$

$$B^T C = \begin{bmatrix} -2 & 0 & 1 \\ 3 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 7 & -3 \\ 1 & -11 & 9 \end{bmatrix}$$

$$A^{-1} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

A is a singular matrix. $\Rightarrow$ No A^{-1}

We say an $n \times n$ matrix A is orthogonal if $A^T A = I_n$. Find x and y in the matrix $A = \begin{bmatrix} \sqrt{3}/2 & x \\ -1/2 & y \end{bmatrix}$ so that A is an orthogonal matrix. (10 pts)

$A = \begin{bmatrix} \sqrt{3}/2 & x \end{bmatrix}$ $\Rightarrow$ Want $A^T \cdot A = I$.

We say an $n \times n$ matrix A is orthogonal if $A^T A = I_n$. Find x and y in the matrix $A = \begin{bmatrix} \sqrt{3}/2 & x \\ -1/2 & y \end{bmatrix}$ so that A is an orthogonal matrix. (10 pts)

$A = \begin{bmatrix} \sqrt{3}/2 & x \\ -1/2 & y \end{bmatrix} \Rightarrow$ Want $A^T \cdot A = I$.

$\Rightarrow \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ x & y \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & x \\ -1/2 & y \end{bmatrix}}_{} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y & x^2+y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1. \end{bmatrix}$

$\Rightarrow \frac{\sqrt{3}}{2}x - \frac{1}{2}y = 0 \Rightarrow y = \sqrt{3}x.$

and $x^2 + y^2 = 1$

$x^2 + (\sqrt{3}x)^2 = 1$

$4x^2 = 1 \Rightarrow x = \pm\frac{1}{2} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

Using determine to determine what value(s) k so that the matrix $A = \begin{bmatrix} k & -k & 3 \\ 0 & (k+1) & 1 \\ k & -8 & (k-1) \end{bmatrix}$ so that A is invertible. (10 pts)

for A to be invertible $\Rightarrow \det(A) \neq 0.$

$\det(A) = k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix}.$

$= k\left[(k+1)(k-1)+8\right] + k\left[-k-3(k+1)\right]$

$= k\left[k^2-1+8\right] + k\,(-4k-3)$

$= k\left[k^2+7-4k-3\right] \neq 0.$

$= k\,(k^2-4k+4) \neq 0.$

$= k\,(k-2)^2 \neq 0 \Rightarrow \underline{k \neq 0, 2}.$