Quiz #3 ***Math 260*** ***10/8/20*** ***Name:*** Key.

Show all your work clearly. No Work, No Credit.

1. Prove or disprove if the following is a subspace of a vector space V. (12pts)

a) $S = \{ f(x) \in P_2 \mid f'(1) + f(2) = 0 \}$

let $y_1, y_2 \in S \Rightarrow \begin{cases} y_1'(1) + y_1(2) = 0 \\ y_2'(1) + y_2(2) = 0 \end{cases} \Rightarrow$ Since $(y_1+y_2)'(1) + (y_1+y_2)(2) = y_1'(1) + y_2'(1) + y_1(2) + y_2(2)$

$= [y_1'(1) + y_1(2)] + [y_2'(1) + y_2(2)]$

$= \underbrace{\quad}_{0} + \underbrace{\quad}_{0} \Rightarrow y_1 + y_2 \in S$ (1)

let $\alpha \in \mathbb{R} \Rightarrow (\alpha y_1)'(1) + (\alpha y_1)(2) =$

$= \alpha y_1'(1) + \alpha y_1(2)$

$= \alpha \underbrace{(y_1'(1) + y_1(2))}$

$= \alpha \cdot 0 \cdot = 0 \Rightarrow \alpha y_1 \in S$ (2)

from (1) & (2) $\Rightarrow$ S is a subspace of P_2

b) $S = \{ A \in M_{3\times3}(R) \mid tr(A) = 2 \}$

$O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow tr(O_{3\times3}) = 0 \Rightarrow O_{3\times3} \notin S.$

$\Rightarrow$ S is not a subspace of $M_{3\times3}$.

c) $S = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3 \mid 2a + b - 3c = 0 \right\}$

let $\vec{V}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ & $\vec{V}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \in S. \Rightarrow \begin{cases} 2a_1 + b_1 - 3c_1 = 0 \\ \text{and} \\ 2a_2 + b_2 - 3c_2 = 0 \end{cases}$

$\vec{V}_1 + \vec{V}_2 = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} \Rightarrow 2(a_1 + a_2) + (b_1 + b_2) - 3(c_1 + c_2)$

$= \underbrace{(2a_1 + b_1 - 3c_1)}_{0} + \underbrace{(2a_2 + b_2 - 3c_2)}_{0} = 0 \Rightarrow \vec{V}_1 + \vec{V}_2 \in S$ (1)

let $\alpha \in \mathbb{R} \Rightarrow \alpha \vec{V}_1 = \begin{bmatrix} \alpha a_1 \\ \alpha b_1 \\ \alpha c_1 \end{bmatrix} \Rightarrow 2(\alpha a_1) + (\alpha b_1) - 3(\alpha c_1)$

$= \alpha [2a_1 + b_1 - 3c_1] = \alpha \cdot 0 = 0 \Rightarrow \alpha \vec{V}_1 \in S$ (2)

from (1) & (2) $\Rightarrow$ S is a subspace of $\mathbb{R}^3$.

d) $S=\left\{A\in M_{2\times 3}(R) \,\middle|\, AA^{T}=I_{2\times 2}\right\}$

$O_{2\times 3}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix} \Rightarrow O_{2\times 3}\cdot O_{2\times 3}^{T} = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\neq I_{2\times 2}.$

$O_{2\times 3}\notin S \Rightarrow S$ is Not a subspace of $M_{2\times 3}$.

2. Determine whether the following set is linearly independent or linearly dependent set. (6 pts)

a) $S=\left\{\begin{bmatrix}3\\-2\\4\end{bmatrix},\begin{bmatrix}1\\-2\\3\end{bmatrix},\begin{bmatrix}5\\2\\0\end{bmatrix}\right\}$

$\qquad\quad \| \quad\ \ \| \quad\ \ \|$
$\qquad\quad V_1 \quad V_2 \quad V_3$

let $\alpha_1 V_1+\alpha_2 V_2+\alpha_3 V_3=0 \Rightarrow \begin{bmatrix}3&1&5\\-2&-2&2\\4&3&0\end{bmatrix} \overset{\text{let}}{=} A$

$\Rightarrow \det(A) = 5\begin{vmatrix}-2&-2\\4&3\end{vmatrix}-2\begin{vmatrix}3&1\\4&3\end{vmatrix}$

$= 5(-6+8)-2(9-4)$

$= 10-10=0$

$\Rightarrow$ S is Linearly Dependent.

b) $S = \{\underbrace{3x^2 - x + 1}_{y_1}, \underbrace{x^2 - 3}_{y_2}, \underbrace{5x + 1}_{y_3}\}$

$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = \alpha_1(3x^2 - x + 1) + \alpha_2(x^2 - 3) + \alpha_3(5x+1) = 0x^2 + 0x + 0$$

$$\Rightarrow (3\alpha_1 + \alpha_2)x^2 + (-\alpha_1 + 5\alpha_3)x + \alpha_1 - 3\alpha_2 + \alpha_3$$

$$\Rightarrow \begin{aligned} 3\alpha_1 + \alpha_2 &= 0 \\ -\alpha_1 + 5\alpha_3 &= 0 \\ \alpha_1 - 3\alpha_2 + \alpha_3 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 5 \\ 1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 0 & 5 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 1 \\ 0 & -3 & 6 \\ 0 & 10 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 10 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 17 \end{bmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\Rightarrow$ The set S is Linearly independent.

3. a) Let $B = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}; \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$. Determine whether or not $Span(B) = R^3$ (4pts)

$$\begin{matrix} \| & \| & \| \\ v_1 & v_2 & v_3 \end{matrix}$$

For B to span $\mathbb{R}^3 \Rightarrow B$ must be linearly independent

$$\Rightarrow \det \begin{bmatrix} -3 & 0 & 2 \\ 2 & 1 & -1 \\ 1 & 11 & 3 \end{bmatrix} = -3 \begin{vmatrix} 1 & -1 \\ 11 & 3 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 1 & 11 \end{vmatrix}$$

$$= -3(3 + 11) + 2(22 - 1)$$

$$= -42 + 42 = 0 \Rightarrow B \text{ is Linear dependent}$$

$\Rightarrow B$ does not span $\mathbb{R}^3$.

$$\overset{y_1}{\overbrace{x^3-x^2}},\quad \overset{y_2}{\overbrace{2x^3-1}},\quad \overset{y_3}{\overbrace{x^2+3x-1}}$$

b) Let $B=\{x^3-x^2,\ 2x^3-1,\ x^2+3x-1\}$. Determine whether $f(x)=-3x^3+2x^2+15x-2\in Span(B)$

(4 pts)

for $f(x)\in Span\{B\} \Rightarrow f(x)=\alpha_1 y_1+\alpha_2 y_2+\alpha_3 y_3$.

$$f(x)=\alpha_1(x^3-x^2)+\alpha_2(2x^3-1)+\alpha_3(x^2+3x-1).$$

$$\Rightarrow -3x^3+2x^2+15x-2=\alpha_1(x^3-x^2)+\alpha_2(2x^3-1)+\alpha_3(x^2+3x-1)$$

$$-3x^3+2x^2+15x-2=(\alpha_1+2\alpha_2)x^3+(-\alpha_1+\alpha_3)x^2+(3\alpha_3)x-\alpha_2-\alpha_3$$

$$\Rightarrow \begin{cases}\alpha_1+2\alpha_2=-3\\ -\alpha_1+\alpha_3=2\\ 3\alpha_3=15\\ \alpha_2-\alpha_3=-2\end{cases}$$

$\alpha_3=5 \Rightarrow -\alpha_1+5=2 \Rightarrow \alpha_1=3.$

$\alpha_2-5=-2 \Rightarrow \alpha_2=3.$

$\Rightarrow f(x)=3\cdot y_1+3y_2+5y_3.$

$\Rightarrow$ Yes $f(x)\in Span\{y_1, y_2, y_3\}$

4. Determine a basis B of the following subspace S of a vector space V, then expand that basis B to a basis of a vector space V. (12 pts)

a) $S=\left\{f(x)\in V=P_2 \,\middle|\, \int_0^1 f(x)\,dx=0\right\}$

Let $f(x)\in S \Rightarrow f(x)=ax^2+bx+c \Rightarrow \int_0^1 f(x)dx=0 \Rightarrow \int_0^1(ax^2+bx+c)dx=\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx\Big|_0^1=\frac{1}{3}a+\frac{1}{2}b+c=0. \Rightarrow c=-\frac{1}{3}a-\frac{1}{2}b.$

$\therefore f(x)=ax^2+bx+c=ax^2+bx-\frac{1}{3}a-\frac{1}{2}b=a(x^2-\frac{1}{3})+b(x-\frac{1}{2}) \Rightarrow span\{x^2-\frac{1}{3}, x-\frac{1}{2}\}$

clearly $B=\{x^2-\frac{1}{3}, x-\frac{1}{2}\}$ is a basis of S.

Since $\dim(P_2)=3 \Rightarrow$ Need one more poly y that is not in $S \Rightarrow y=1$. Since $\int_0^1 1dx=1\neq 0 \Rightarrow y=1\notin S$.

Extended basis. $\{x^2-\frac{1}{3}, x-\frac{1}{2}, 1\}$.

is a basis of P_2.

b) $S=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in M_{2\times 2}(R) \mid a+2b=3c-d\right\} \Rightarrow a = 3c-d-2b.$

so for $A \in S \Rightarrow A = \begin{bmatrix} 3c-d-2b & b \\ c & d \end{bmatrix} = c\begin{bmatrix} 3 & 0 \\ 1 & 0\end{bmatrix} + b\begin{bmatrix} -2 & 1 \\ 0 & 0\end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1\end{bmatrix}$

$\Rightarrow S = \text{span}\left\{\begin{bmatrix} 3 & 0 \\ 1 & 0\end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1\end{bmatrix}\right\}.$

Since $\dim(M_{2\times 2}) = 4 \Rightarrow$ We need one more matrix that is not in S.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}$ b/c $a = 1 = 3(0)-0-2(0) = 0$

$1 \neq 0 . \Rightarrow A \notin S.$

Extended basis: $\left\{\begin{bmatrix} 3 & 0 \\ 1 & 0\end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1\end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}\right\}$

is a basis of $M_{2\times 2}$.

c) $S=\left\{\begin{bmatrix} a \\ b \\ c\end{bmatrix} \in R^3 \mid 2a+b-3c=0\right\} \Rightarrow b = 3c-2a.$

for any $\vec{v} \in S \Rightarrow \vec{v} = \begin{bmatrix} a \\ 3c-2a \\ c\end{bmatrix} = a\begin{bmatrix} 1 \\ -2 \\ 0\end{bmatrix} + c\begin{bmatrix} 0 \\ 3 \\ 1\end{bmatrix}.$

$\Rightarrow S = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 0\end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1\end{bmatrix}\right\}.$

$\dim(\mathbb{R}^3) = 3 \Rightarrow$ Need one more vector extended from a basis of S.

$\vec{V} = \begin{bmatrix} 1 \\ 0 \\ 0\end{bmatrix}$ b/c $2(1)+0-3(0) = 2 \neq 0.$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0\end{bmatrix} \notin S.$

$\Rightarrow$ Extended basis: $\left\{\begin{bmatrix} 1 \\ -2 \\ 0\end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1\end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^3$.

5. Find a minimal set of vectors of B that spans the same vector space. (8 pts)

a) $B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}; \begin{bmatrix} -4 \\ -1 \\ -6 \end{bmatrix}; \begin{bmatrix} 7 \\ 1 \\ 9 \end{bmatrix} \right\}$

Let $\Rightarrow \vec{V}_1 \quad \vec{V}_2 \quad \vec{V}_3 \quad \vec{V}_4$

Set $\alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 + \alpha_3 \vec{V}_3 + \alpha_4 \vec{V}_4 = \vec{0}$.

$$\begin{bmatrix} -2 & 1 & -4 & 7 \\ 1 & -2 & -1 & 1 \\ 0 & -3 & -6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 1 \\ -2 & 1 & -4 & 7 \\ 0 & -3 & -6 & 9 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & -3 & -6 & 9 \\ 0 & -3 & -6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \alpha_2 + 2\alpha_3 - 3\alpha_4 = 0$$

$\alpha_2 = 3\alpha_4 - 2\alpha_3$.

and $\alpha_1 - 2(3\alpha_4 - 2\alpha_3) - \alpha_3 + \alpha_4 = 0$

$\alpha_1 - 6\alpha_4 + 4\alpha_3 - \alpha_3 + \alpha_4 = 0$

$\alpha_1 = 5\alpha_4 - 3\alpha_3$.

$\Rightarrow$ Clearly $\vec{V}_1$ & $\vec{V}_2$ can be expressed as a linear Combination of $\vec{V}_3$ & $\vec{V}_4$

$\Rightarrow$ minimal set of s $\{\vec{V}_3, \vec{V}_4\}$.

b) $B = \{3x - x^2; 1 + 3x; 3 + 15x - 2x^2\}$

$\quad\; y_1 \qquad y_2 \qquad y_3$.

$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = \alpha_1 (3x - x^2) + \alpha_2 (1 + 3x) + \alpha_3 (3 + 15x - 2x^2) = 0.$$

$$(-\alpha_1 - 2\alpha_3)x^2 + (3\alpha_1 + 3\alpha_2 + 15\alpha_3)x + \alpha_2 + 3\alpha_3 = 0.$$

$$\Rightarrow \begin{cases} -\alpha_1 - 2\alpha_3 = 0 \\ 3\alpha_1 + 3\alpha_2 + 15\alpha_3 = 0 \\ \alpha_2 + 3\alpha_3 = 0 \end{cases} \Rightarrow \begin{bmatrix} -1 & 0 & -2 \\ 3 & 3 & 15 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \rightarrow \alpha_2 = -3\alpha_3 \\ \rightarrow -\alpha_1 = 2\alpha_3 \Rightarrow \alpha_1 = -2\alpha_3 \end{matrix}$$

$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = -2\alpha_3 y_1 - 3\alpha_3 y_2 + \alpha_3 y_3 = 0$$

$$\Rightarrow -2y_1 - 3y_2 + y_3 = 0$$

$\rightarrow y_3 = 2y_1 + 3y_2$.

$\Rightarrow$ drop y_3

$\Rightarrow$ minimal set $\{y_1, y_2\}$

6. Let V be a vector space and $B = \left\{ \vec{v}_1 ; \vec{v}_2 ; \ldots \vec{v}_n \middle| \vec{v}_i \in V;\ for\ i = 1,2,3,\ldots,n \right\}$. Prove that Span(B) is a subspace of V. (4pts)

Lecture notes.