Quiz #4 **Math 260** **4/25/20** **Name:** KEY.

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1. Prove or disprove (disprove = counter examples) if the following is a subspace of a vector space V. (12pts)

a) $$S=\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}\in R^3 \,\middle|\, 2a+3b-c=0\right\}$$

Let $\vec{V}_1=\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}\in S$ & $\vec{V}_2=\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}\in S \Rightarrow \begin{cases}2a_1+3b_1-c_1=0\\ 2a_2+3b_2-c_2=0.\end{cases}$

$$\Rightarrow \vec{V}_1+\vec{V}_2=\begin{bmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\end{bmatrix}\quad 2(a_1+a_2)+3(b_1+b_2)-(c_1+c_2)\overset{?}{=}0$$

$$\underbrace{(2a_1+3b_1-c_1)}_{\overset{\shortparallel}{0}}+\underbrace{(2a_2+3b_2-c_2)}_{0}\overset{?}{=}0$$

$$\overset{\checkmark}{=}0 \Rightarrow \vec{V}_1+\vec{V}_2\in S \quad (1)$$

let $\alpha\in\mathbb{R}\Rightarrow \alpha\vec{V}_1=\begin{bmatrix}\alpha a_1\\\alpha b_1\\\alpha c_1\end{bmatrix}\Rightarrow (\alpha a_1)+3(\alpha b_1)-\alpha c_1\overset{?}{=}0$

$$\alpha\underbrace{(a_1+3b_1-c_1)}_{\overset{\shortparallel}{0}}\overset{?}{=}0$$

$$\alpha\cdot 0\ \overset{\checkmark}{=}0 \Rightarrow \alpha\vec{V}_1\in S \quad (2)$$

from (1) & (2) $\Rightarrow$ S is a subspace of $\mathbb{R}^3$

b) $$S=\left\{A\in M_{2\times2}\,\middle|\,2A=A^T\right\}=\left\{A\in M_{2\times2}\,\middle|\,2A-A^T=O_{2\times2}\right\}.$$

let A_1 & $A_2\in S\Rightarrow\begin{cases}2A_1-2A_1^T=O_2\\ 2A_2-2A_2^T=O_2.\end{cases}$

$$A_1+A_2\Rightarrow 2(A_1+A_2)-(A_1+A_2)^T\overset{?}{=}O_2.$$

$$2A_1+2A_2-(A_1^T+A_2^T)\overset{?}{=}O_2$$

$$\underbrace{(2A_1-A_1^T)}_{\overset{\shortparallel}{O_2}}+\underbrace{(2A_2-A_2^T)}_{\overset{\shortparallel}{O_2}}\overset{?}{=}O_2$$

$$\overset{\checkmark}{=}O_2\Rightarrow A_1+A_2\in S.$$

let $\alpha\in\mathbb{R}\Rightarrow \alpha A_1\overset{?}{\in}S \Rightarrow 2(\alpha A_1)-(\alpha A_1)^T\overset{?}{=}O_2.$

$$2\alpha A_1-\alpha A_1^T\overset{?}{=}O_2.$$

$$\alpha\underbrace{(2A_1-A_1^T)}_{\overset{\shortparallel}{O_2}}\overset{?}{=}O_2$$

$$\alpha\cdot O_2\ \overset{\checkmark}{=}O_2\Rightarrow\alpha A_1\in S$$

$\Rightarrow$ S is a subspace of $M_{2\times2}(\mathbb{R})$

c) $S = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3 \middle| \int_0^2 (ax^2 + bx + c)\, dx = 0 \right\}$

Let $\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \in S \Rightarrow \begin{cases} \int_0^2 (a_1x^2 + b_1x + c_1)\, dx = 0 \\ \& \\ \int_0^2 (a_2x^2 + b_2x + c_2)\, dx = 0 \end{cases}$

clearly : $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} \Rightarrow \int_0^2 \left[(a_1 + a_2) x^2 + (b_1 + b_2) x + (c_1 + c_2) \right] dx \overset{?}{=} 0.$

$\Rightarrow \underbrace{\int_0^2 (a_1x^2 + b_1x + c_1)\, dx}_{=0} + \underbrace{\int_0^2 (a_2x^2 + b_2x + c_2)\, dx}_{=0} \overset{\checkmark}{=} 0$

$\Rightarrow \vec{v}_1 + \vec{v}_2 \in S.$

& $\alpha \vec{v}_1 = \begin{bmatrix} \alpha a_1 \\ \alpha b_1 \\ \alpha c_1 \end{bmatrix} \overset{?}{\in} S \Rightarrow \int_0^2 (\alpha a_1 x^2 + \alpha b_1 x + \alpha c_1)\, dx = \alpha \underbrace{\int_0^2 (a_1x^2 + b_1x + c_1)\, dx}_{} = \alpha \cdot 0 \overset{\checkmark}{=} 0$

$\Rightarrow$ Yes, S is a subspace of S.

d) $S = \{ f(x) \in P_2 \mid f(2) + f'(0) = 1 \}$

let $f(x) = 1 \Rightarrow f(2) + f'(0) = 1 + 0 = 1 \Rightarrow f(x) \in S.$

let $\alpha = 2 \Rightarrow g(x) = \alpha f(x) = 2 \cdot 1 = 2.$

$g(2) + g'(0) = 2 + 0 = 2 \neq 1 \Rightarrow \alpha \cdot f(x) \notin S.$

$\Rightarrow S$ is not a subspace of P_2.

2. a) Let $B=\{\overbrace{3x^2-2x+1}^{f_1},\ \overbrace{x^2-2x}^{f_2},\ \overbrace{3x^2+2x+2}^{f_3}\}$. Determine whether or not $Span(B)=P_2$ (4pts)

Let $f(x)=ax^2+bx+c=\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3$.

$$=\alpha_1(3x^2-2x+1)+\alpha_2(x^2-2x)+\alpha_3(3x^2+2x+2)$$

$$=(3\alpha_1+\alpha_2+3\alpha_3)x^2+(-2\alpha_1-2\alpha_2+2\alpha_3)x+\alpha_1+2\alpha_3$$

$$\Rightarrow \begin{cases} 3\alpha_1+\alpha_2+3\alpha_3=a. \\ -2\alpha_1-2\alpha_2+2\alpha_3=b \\ \alpha_1+2\alpha_3=c \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 3 & 1 & 3 & a \\ -2 & -2 & 2 & b \\ 1 & 0 & 2 & c \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & c \\ -2 & -2 & 2 & b \\ 3 & 1 & 3 & a \end{array}\right]$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 2 & c \\ 0 & -2 & 6 & b+2c \\ 0 & 1 & -3 & a-3c \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & c \\ 0 & 1 & -3 & a-3c \\ 0 & -2 & 6 & b+2c \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & c \\ 0 & 1 & -3 & a-3c \\ 0 & 0 & 0 & 2a+b-4c \end{array}\right]$$

So for any $f(x)=ax^2+bx+c\in Span\{B\}\Rightarrow 2a+b-4c=0$.

Then clearly $f(x)=1\notin span(B)\Rightarrow B$ does not span P_2.

b) Let $B=\left\{\underset{\overset{\|}{v_1}}{\begin{bmatrix}-1\\0\\1\\3\end{bmatrix}},\underset{\overset{\|}{v_2}}{\begin{bmatrix}2\\-1\\3\\0\end{bmatrix}},\underset{\overset{\|}{v_3}}{\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}}\right\}$. Determine whether $\vec{v}=\begin{bmatrix}-1\\-8\\11\\11\end{bmatrix}\in span(B)$ (4 pts)

Consider $\vec{v}=\alpha_1\vec{v_1}+\alpha_2\vec{v_2}+\alpha_3\vec{v_3}\Rightarrow \left[\begin{array}{ccc|c} -1 & 2 & 1 & -1 \\ 0 & -1 & 1 & -8 \\ 1 & 3 & 0 & 11 \\ 3 & 0 & -1 & 11 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 2 & 1 & -1 \\ 0 & -1 & 1 & -8 \\ 0 & 5 & 1 & 10 \\ 0 & 6 & 2 & 8 \end{array}\right]$

$$=\left[\begin{array}{ccc|c} -1 & 2 & 1 & -1 \\ 0 & -1 & 1 & -8 \\ 0 & 0 & 6 & -30 \\ 0 & 0 & 8 & -40 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 2 & 1 & -1 \\ 0 & -1 & 1 & -8 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$\Rightarrow$ Clearly $\alpha_3=-5$, $-\alpha_2-5=-8\Rightarrow\alpha_2=3$. and $-\alpha_1+2(3)-5=-1$
$\alpha_1=2$.

$\therefore\ \vec{v}=2\vec{v_1}+3\vec{v_2}-5v_3.\Rightarrow$ Yes, $\vec{v}\in span\{B\}$.

3. Determine a basis B of the following subspace S of a vector space V, then expand that basis B to a basis of a vector space V. (8 pts)

a) $S = \{f(x) \in P_2 \mid f(1) + f'(1) = 0\}$

let $f(x) = ax^2 + bx + c \in S \Rightarrow f(1) = 2a + b + c$.

$+\ f'(1) = 2a + b$

$4a + 2b + c = 0 \Rightarrow c = -4a - 2b$.

$\Rightarrow f(x) = ax^2 + bx - 4a - 2b := a(x^2 - 4) + b(x - 2)$

$\Rightarrow S = \text{span}\{x^2 - 4,\ x - 2\} \Rightarrow$ Basis of $S = \{\underbrace{x^2 - 4,\ x - 2}_{\Rightarrow \text{ clearly this is L.I.}}\}$.

to expand to a basis of $P_2 \Rightarrow$ Need $c \neq -4a - 2b$.

choose $c = 1,\ a = 0,\ b = 0$.

$\Rightarrow f(x) = 1$.

$\Rightarrow$ Expanded basis for P_2.

$\{x^2 - 4,\ x - 2,\ 1\}$.

b) $S = \{A \in M_{2\times 2} \mid A^T = A\}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \Rightarrow A^T = A \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \left\{\begin{matrix} a = a;\ b = c \\ c = b;\ d = d \end{matrix}\right\} \Rightarrow b = c$.

So $A \in S \Rightarrow A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$S = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

Since $\dim(M_{2\times 2}) = 4 \Rightarrow$ Need one more that requires $c \neq d$. $\Rightarrow$ pick $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Expanded basis for $M_{2\times 2}(\mathbb{R}) = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$.

4. Find a minimal set of vectors of B that spans the same vector space. (8 pts)

a) $B=\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix},\begin{bmatrix}-3\\1\\7\end{bmatrix},\begin{bmatrix}-3\\2\\-1\end{bmatrix},\begin{bmatrix}-2\\0\\10\end{bmatrix}\right\}$

$\quad\quad\quad\quad \underset{v_1}{\shortparallel} \quad \underset{v_2}{\shortparallel} \quad \underset{v_3}{\shortparallel} \quad \underset{v_4}{\shortparallel} \Rightarrow$ clearly $v_4=\vec{v_1}+\vec{v_2} \Rightarrow$ drop $\vec{v_4}$

$\Rightarrow \{v_1,v_2,v_3\} \Rightarrow \det\begin{bmatrix}1&-3&-3\\-1&1&2\\3&7&-1\end{bmatrix}=\begin{vmatrix}1&2\\7&-1\end{vmatrix}+3\begin{vmatrix}-1&2\\3&-1\end{vmatrix}-3\begin{vmatrix}-1&1\\3&7\end{vmatrix}=(-1-14)+3(1-6)-3(-7-3)$

$= -15-15+30=0 \Rightarrow \{\vec{v_1},\vec{v_2},\vec{v_3}\}$ is L.D.

drop $\vec{v_1}$ = $\{\vec{v_2},\vec{v_3}\}=\left\{\begin{bmatrix}-3\\1\\7\end{bmatrix},\begin{bmatrix}-3\\2\\1\end{bmatrix}\right\}$ is clearly L.I. {Not multiple of each other}

$\Rightarrow$ minimal set $\left\{\begin{bmatrix}-3\\1\\7\end{bmatrix},\begin{bmatrix}-3\\2\\1\end{bmatrix}\right\}$

b) $B=\{\overbrace{3x-2x^2}^{y_1},\overbrace{1-5x-4x^2}^{y_2},\overbrace{9x-1}^{y_3}\}$

Want to determine if $\{y_1,y_2,y_3\}$ is L.I. $\Rightarrow \alpha_1y_1+\alpha_2y_2+\alpha_3y_3=0$ if $\alpha_1=\alpha_2=\alpha_3=0$.

$\Rightarrow \alpha_1(3x-2x^2)+\alpha_2(1-5x-4x^2)+\alpha_3(9x-1)=0$

$(-2\alpha_1-4\alpha_2)x^2+(3\alpha_1-5\alpha_2+9\alpha_3)x+\alpha_2-\alpha_3=0x^2+0x+0.$

$$\begin{cases}-2\alpha_1-4\alpha_2=0 \Rightarrow \alpha_1=2\alpha_2.\\ 3\alpha_1-5\alpha_2+9\alpha_3=0 \Rightarrow 3\cdot2\alpha_2-5\alpha_2+9\alpha_3=0 \Rightarrow \alpha_2+9\alpha_3=0\\ \alpha_2-\alpha_3=0\end{cases}$$

$\alpha_2-\alpha_3=0$

$10\alpha_3=0 \Rightarrow \alpha_3=0 \Rightarrow \alpha_2=0 \Rightarrow \alpha_3=0$

$\Rightarrow \{y_1,y_2,y_3\}$ is L.I.

$\Rightarrow$ minimal set of $B=\{y_1,y_2,y_3\}$.

5. Let $B=\{\vec{v_1},\vec{v_2},\vec{v_3},\ldots,\vec{v_n}\}$ be a basis of a finite dimension vector space V. Prove that for every $\vec{v}\in V$ there is only one way to express $\vec{v}\in V$ as a linear combination vectors of B. (4pts)

Let $\vec{v}\in V \Rightarrow \vec{v}\in \text{span}(B)$.

Suppose $\vec{v}$ has 2 linear combination of B.

Say: $\vec{v}=\alpha_1\vec{v_1}+\alpha_2\vec{v_2}+\alpha_3\vec{v_3}+\cdots+\alpha_n\vec{v_n}$

and $\vec{v}=\beta_1\vec{v_1}+\beta_2\vec{v_2}+\beta_3\vec{v_3}+\cdots+\beta_n\vec{v_n}$

$\vec{0}=(\alpha_1-\beta_1)\vec{v_1}+(\alpha_2-\beta_2)\vec{v_2}+(\alpha_3-\beta_3)\vec{v_3}+\cdots+(\alpha_n-\beta_n)\vec{v_n}=0$

Since B is a basis, $\{\vec{v_1},\ldots,\vec{v_n}\}$ is L.I.

$\Rightarrow \alpha_1-\beta_1=0 \Rightarrow \alpha_1=\beta_1$

$\alpha_2-\beta_2=0 \Rightarrow \alpha_2=\beta_2$

$\alpha_3-\beta_3=0 \Rightarrow \alpha_3=\beta_3$

$\vdots$

$\alpha_n-\beta_n=0 \Rightarrow \alpha_n=\beta_n$.

$\Rightarrow$ Linear combination of v is unique.

6. Determine the rank(A), nullity of A, basis of Nullspace(A) and basis of Colspace(A). (10 pts)

a) $$\begin{array}{c} 3 \\ \\ 2 \end{array} A=\begin{bmatrix}2&-1&3&0\\-6&-4&21&20\\-3&0&3&5\end{bmatrix} = \begin{array}{c} \\ -3 \\ 7 \end{array}\begin{bmatrix}2&-1&3&0\\0&-7&30&20\\0&-3&15&10\end{bmatrix}=\begin{bmatrix}2&-1&3&0\\0&-7&30&20\\0&0&15&10\end{bmatrix}=\begin{bmatrix}2&-1&3&0\\0&-7&30&20\\0&0&3&2\end{bmatrix}$$

$$\text{rowspace}=\text{span}\left\{\begin{bmatrix}2\\-1\\3\\0\end{bmatrix},\begin{bmatrix}0\\-7\\30\\20\end{bmatrix},\begin{bmatrix}0\\0\\3\\2\end{bmatrix}\right\} \quad \& \quad \text{colspace}(A)=\left\{\begin{bmatrix}2\\-6\\-3\end{bmatrix},\begin{bmatrix}-1\\-4\\0\end{bmatrix},\begin{bmatrix}3\\21\\3\end{bmatrix}\right\}$$

$\Rightarrow \text{Rank}(A)=3. \Rightarrow \text{Nullity}(A)+\text{Rank}(A)=4,$

$\Rightarrow \text{Nullity}(A)=4-3=1.$

Basis of Nullspace(A) $\Rightarrow 3z+2w=0 \Rightarrow z=-\frac{2}{3}w=-\frac{2}{3}t \ \{\text{let } w=t\}$

$-7y+30z+20w=0 \Rightarrow -7y+30\left(-\frac{2}{3}t\right)+20t=0$

$-7y-20t+20t \Rightarrow y=0.$

$2x-y+3z=0 \Rightarrow 2x-0+3\left(-\frac{2}{3}t\right)=0 \Rightarrow x=t$

$$\Rightarrow \vec{v}=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\in \text{Nullspace}(A) \Rightarrow \vec{v}=\begin{bmatrix}t\\0\\-2/3\,t\\t\end{bmatrix}=\frac{1}{3}t\begin{bmatrix}3\\0\\-2\\3\end{bmatrix} \text{ for } \forall t\in\mathbb{R}.$$

$$\text{Nullspace}(A)=\text{span}\left\{\begin{bmatrix}3\\0\\-2\\3\end{bmatrix}\right\} \Rightarrow \text{Basis of Nullspace}(A)=\left\{\begin{bmatrix}3\\0\\-2\\3\end{bmatrix}\right\}$$

$\begin{matrix} 2 & 2 \\ 2 & 5 \end{matrix}$

b) $A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 2 & -4 \\ 3 & 21 & -3 \\ -4 & 18 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 8 & -2 \\ 0 & 12 & -6 \\ 0 & 30 & 3 \end{bmatrix} = \begin{matrix} 5 \\ \\ -2 \end{matrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & -1 \\ 0 & 2 & -1 \\ 0 & 10 & 1 \end{bmatrix} \overset{-2}{=} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 4 & -1 \\ 0 & 10 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of Rowspace $(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \text{Rank}(A) = 3.$

Basis of Colspace $(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 21 \\ 18 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -3 \\ -1 \end{bmatrix} \right\}$

$\text{Nullity}(A) + \text{Rank}(A) = 3. \Rightarrow \text{Nullity}(A) = 0.$

Basis of Nullspace $(A) = \left\{ \right.$ Third row from REF(A) $\Rightarrow z = 0$

$\Rightarrow y = 0 \Rightarrow \text{Nullspace}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow x = 0.$