Quiz #5 **Math 260** **Name:** KEY

11/5/20

Show all your work clearly. No Work, No Credit.

1. Determine the component vector $\left[\vec{v}\right]_B$ *or* $\left[p(x)\right]_B$ relative to the given ordered basis B. (8 pts)

a) $V=R^3;\ B=\left\{\underset{\overset{\shortparallel}{v_1}}{\begin{bmatrix}1\\-6\\3\end{bmatrix}},\underset{\overset{\shortparallel}{v_2}}{\begin{bmatrix}0\\5\\-1\end{bmatrix}},\underset{\overset{\shortparallel}{v_3}}{\begin{bmatrix}3\\-1\\-1\end{bmatrix}}\right\},\ \vec{v}=\begin{bmatrix}1\\7\\7\end{bmatrix}$

$$\vec{V}=\alpha_1\vec{V}_1+\alpha_2\vec{V}_2+\alpha_3\vec{V}_3 \Rightarrow \begin{array}{r}-3\\ \\ \\ \end{array}\left[\begin{array}{ccc|c}1&0&3&1\\-6&5&-1&7\\3&-1&-1&7\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1&0&3&1\\0&5&17&13\\0&-1&-10&4\end{array}\right]=_5\left[\begin{array}{ccc|c}1&0&3&1\\0&-1&-10&4\\0&5&17&13\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}1&0&3&1\\0&-1&-10&4\\0&0&-33&33\end{array}\right]\rightarrow \begin{aligned}&\alpha_3=-1\\&-\alpha_2=4+10(-1)=-6\\&\alpha_2=6.\\&\alpha_1=1-3(-1)=4.\end{aligned}$$

$$\Rightarrow \left[V\right]_B=\begin{bmatrix}4\\6\\-1\end{bmatrix}$$

b) $V = P_2;\ B = \{\underbrace{x^2+x}_{P_1}, \underbrace{2+2x}_{P_2}, \underbrace{1}_{P_3}\}\ p(x) = -4x^2+2x+6$

$$p(x) = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 = \alpha_1 (x^2+x) + \alpha_2 (2+2x) + \alpha_3$$

$$-4x^2 + 2x + 6 = \alpha_1 x^2 + (\alpha_1 + 2\alpha_2)x + 2\alpha_2 + \alpha_3$$

$$\Rightarrow \alpha_1 = -4;\quad \alpha_1 + 2\alpha_2 = 2$$

$$-4 + 2\alpha_2 = 2 \Rightarrow \alpha_2 = 3$$

$$2\alpha_2 + \alpha_3 = 6$$

$$2(3) + \alpha_3 = 6$$

$$\alpha_3 = 0$$

$$\Rightarrow \left[p(x)\right]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

2. Find the change – of – basis matrix $[P]_{B\to C}$ for vector space (7 pts)

$$V = R^3;\ B = \left\{ \underbrace{\begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 8 \\ -2 \\ -9 \end{bmatrix}}_{v_3} \right\} \text{ and } C = \left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}}_{u_3} \right\}$$

$$[P]_{B\to C} = \begin{bmatrix} | & | & | \\ [v_1]_c & [v_2]_c & [v_3]_c \\ | & | & | \end{bmatrix}$$

$$\Rightarrow {}^{-1}\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 3 & 8 \\ -1 & 1 & 1 & -5 & 0 & -2 \\ 1 & 1 & 3 & 0 & 5 & -9 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} \\ -\cdot 1 \\ \\ \end{array}\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 3 & 8 \\ 0 & 1 & 1 & -3 & 3 & 6 \\ 0 & 1 & 3 & 2 & 8 & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 3 & 8 \\ 0 & 1 & 1 & -3 & 3 & 6 \\ 0 & 0 & 2 & 5 & 5 & -7 \end{array}\right]$$

for $\vec{v}_1 \Rightarrow \begin{cases} 2\alpha_3 = 5 \Rightarrow \alpha_3 = 5/2 \\ \alpha_2 = -3 - \frac{5}{2} = -11/2. \\ \alpha_1 = 2 \end{cases}$

for $v_2 \Rightarrow 2\alpha_3 = 5 \Rightarrow \alpha_3 = 5/2$

$\alpha_2 + \alpha_3 = 3 \Rightarrow \alpha_2 = 3 - \frac{5}{2} = 1/2$

$\alpha_1 = 3$

for $\vec{v}_3 \Rightarrow \begin{cases} 2\alpha_3 = -7 \Rightarrow \alpha_3 = -7/2 \\ \alpha_2 = 6 - \alpha_3 = 6 - \frac{7}{2} = 5/2 \\ \alpha_1 = 8 \end{cases}$ $\Rightarrow [P]_{B\to C} = \begin{bmatrix} 2 & 3 & 8 \\ -\frac{11}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} & -\frac{7}{2} \end{bmatrix}.$

OR $[P]_{B\to C} = \frac{1}{2}\begin{bmatrix} 4 & 6 & 16 \\ -11 & 1 & 5 \\ 5 & 5 & -7 \end{bmatrix}$

3. a) Define a mapping $\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = 2v_1w_1 + v_1w_2 + v_2w_1 + 2v_2w_2$ for all (5pts)

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in V = R^2$. Prove that V and <,> is an inner product vector space.

1) Prove $\langle \vec{v}, \vec{v} \rangle \geq 0 \Rightarrow \left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = 2v_1^2 + v_1v_2 + v_2v_1 + 2v_2^2$

$$= v_1^2 + 2v_1v_2 + v_2^2 + v_1^2 + v_2^2$$

$$= (v_1 + v_2)^2 + v_1^2 + v_2^2 \geq 0$$

Clearly $\langle v, v \rangle = 0 \Rightarrow v_1 = v_2 = 0$

2) $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$

$$\langle v, w \rangle = 2v_1w_1 + v_1w_2 + v_2w_1 + 2v_2w_2$$

$$= 2w_1v_1 + w_2v_1 + w_2v_2 + 2w_2v_2 = \langle w, v \rangle$$

3) $\langle \alpha v, w \rangle = 2\alpha v_1w_1 + \alpha v_1w_2 + \alpha v_2w_1 + 2\alpha v_2w_2$

$$= \alpha(2v_1w_1 + v_1w_2 + v_2w_1 + 2v_2w_2)$$

$$= \alpha \langle v, w \rangle$$

4) $\langle \vec{v}, \vec{u} + \vec{w} \rangle \overset{?}{=} \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{w} \rangle.$

$$\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \end{bmatrix} \right\rangle = 2v_1(u_1 + w_1) + v_1(u_2 + w_2) + v_2(u_2 + w_2) + 2v_2(u_2 + w_2)$$

$$= \underbrace{2v_1u_1 + v_1u_2 + v_2u_2 + 2v_2u_2}_{\langle \vec{v}, \vec{u} \rangle} + \underset{+ \ \langle \vec{v}, \vec{w} \rangle}{2v_1w_1 + v_1w_2 + v_2w_2 + 2v_2w_2}$$

$$=$$

b) Let $\vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Using the mapping in part (a) to determine $proj_{\vec{w}} \vec{v}$ (5pts)

$$\text{proj}_{\vec{w}} \vec{v} = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \cdot \vec{w} .$$

$$\left\langle \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle = 2(-2)(1) + (-2)(2) + (3)(1) + 2(3)(2)$$

$$= -4 - 4 + 3 + 12 = 7 .$$

$$\langle w, w \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle = 2(1)(1) + 1(2) + 2(1) + 2(2)(2)$$

$$= 2 + 2 + 2 + 8 = 14 .$$

$$\text{proj}_{\vec{w}} \vec{v} = \frac{7}{14} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} . = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} .$$

4. Use the Gram – Schmidt process to generate an orthogonal basis from the following basis.

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \right\}$$ for $V = R^3$ as standard inner product vector space. (5 pts)

$$\underset{V_1}{\shortparallel} \quad \underset{V_2}{\shortparallel} \quad \underset{V_3}{\shortparallel}$$

$$\vec{u}_1 = \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \; ; \quad \vec{u}_2 = \vec{V}_2 - \text{proj}_{\vec{V}_2}^{\vec{u}_1} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{\langle \vec{u}_1, \vec{V}_2 \rangle}{\|\vec{V}_2\|^2} \cdot \vec{V}_2$$

where $\langle u_1, \vec{V}_2 \rangle = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 1(0) + 1(2) + (0)(1) = 2.$

$\|\vec{V}_2\|^2 = \left\langle \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 0 + 4 + 1 = 5.$

$\Bigg\}\ \text{proj}_{V_2}^{u} = \frac{2}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}. = \begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix}.$

$$\Rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 - 4/5 \\ 1 - 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix}$$

$$\vec{u}_2 = \vec{V}_3 - \text{proj}_{\vec{u}_1}^{\vec{V}_3} - \text{proj}_{\vec{u}_2}^{\vec{V}_3} = V_3 - \frac{\langle \vec{V}_3, \vec{u}_1 \rangle}{\|u_1\|^2} \cdot \vec{u}_1 - \frac{\langle \vec{V}_3, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \cdot \vec{u}_2$$

$\langle \vec{V}_3, \vec{u}_1 \rangle = \left\langle \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle = 1 + 1 = 2. \; ; \quad \langle \vec{V}_3, \vec{u}_2 \rangle = \left\langle \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix} \right\rangle = \frac{6}{5} + \frac{18}{5} = \frac{24}{5}$

$\|u_1\|^2 = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle = 1 + 1 = 2 \; ; \quad \|\vec{u}_2\|^2 = \left\langle \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix} \right\rangle = \frac{36}{25} + \frac{9}{25} = \frac{45}{25} = \frac{9}{5}$

$$\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{24/\cancel{5}}{9/\cancel{5}} \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left[\; 0 \right.$$

$$= \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{7}{3} \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 42/15 \\ 7/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -42/5 \\ 23/5 \end{bmatrix}$$

Orthogonal basis: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6/5 \\ 3/5 \end{bmatrix}, \begin{bmatrix} 0 \\ -42/5 \\ 23/5 \end{bmatrix} \right\}$

5. Consider a basis $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ for R^2. Find a linear transformation $T : R^2 \mapsto R^3$ such that

$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ *and* $T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (5pts)

Given any vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

$\Rightarrow \alpha_1 = x$ & $-\alpha_1 + 3\alpha_2 = y$

$-x + 3\alpha_2 = y \Rightarrow \alpha_2 = \frac{x+y}{3}$

So for any $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{x+y}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

$$\Rightarrow T(\vec{v}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{x+y}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$= x \underbrace{T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)} + \frac{x+y}{3} T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right),$$

$$= x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{x+y}{3} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} x + \frac{2}{3}(x+y) \\ -x + \frac{x+y}{3} \\ \frac{x+y}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{3}x + \frac{2}{3}y \\ -\frac{2}{3}x + \frac{1}{3}y \\ \frac{1}{3}x + \frac{1}{3}y \end{bmatrix}$$

6. Determine a basis of Ker(T) and Rng(T) of the following linear transformation, then determine whether T is injective, surjective or bijective. (15 pts)

a) $T : R^3 \mapsto P_2 \text{ by } T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a-b)x^2 + (2b+c)x + 3a - b + c$

$$\ker(T) = \left\{ \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a-b)x^2 + (2b+c)x + 3a-b+c = 0 \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} a-b=0 \\ 2b+c=0 \\ 3a-b+c=0 \end{array} \right\} \Rightarrow \begin{array}{c} -3 \\ \\ \\ \end{array}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} 2b+c=0 \\ c=-2b \\ a-b=0 \Rightarrow a=b \end{array}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker(T) \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ b \\ -2b \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}$$

$$\text{Rng}(T) = \left\{ p(x) = ax^2 + bx + c \in P_2 \;\middle|\; \exists\, \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \text{ such tat } T(v) = p(x) \right\}$$

$$\Rightarrow T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a-b)x^2 + (2b+c)x + 3a-b+c$$

$$= a(x^2+3) + b(-x^2+2x-1) + c\,(x+1)$$

$$= \text{Span}\{x^2+3,\ -x^2+2x-1,\ x+1\}$$

Since Nullity $(T) = 1 \Longrightarrow \dim(\text{Rng}(T))$ must be true.

$$\Rightarrow \text{Rng}(T) = \text{span}\{x^2+3,\ -x^2+2x-1\}$$

b) $T : P_2 \mapsto M_2(R)$ by $T(ax^2+bx+c)=\begin{bmatrix} a+b & b+c \\ 2c-a & a-c \end{bmatrix}$

$\ker(T) = \{ p(x) \in P_2 \mid T(p(x)) = 0 \in M_2(R) \}$

$\Rightarrow p(x) = ax^2+bx+c \Rightarrow T(p(x)) = \begin{bmatrix} a+b & b+c \\ 2c-a & a-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$

$a+b=0 \Rightarrow b=-a.$

$b+c=0$

$2c-a=0$

$a-c=0$

$c=0 \Rightarrow a=0,\ b=0.$

$\ker(T) = \{0x^2+0x+0\}.$

$\Rightarrow \text{Nullity}(T)=0$

$\text{Rng}(T) = \left\{ A = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \in M_2 \;\middle|\; \exists\, p(x) \in P_2 \mid T(p(x)) = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \right\}$

$\Rightarrow T(p) = \begin{bmatrix} a+b & b+c \\ 2c-a & a-c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$

$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \right\}$

Since $\text{Nullity}(T)=0 \Rightarrow \text{Rank}(T)=3$

$\Rightarrow \text{Rng}(T) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \right\}$

c) $T : P_3 \mapsto P_2$ by $T(f(x)) = 2f'' + f'$

$$\ker(T) = \{ f(x) = ax^3 + bx^2 + cx + d \in P_3 \mid T(f(x)) = 0 = 0x^2 + 0x + 0 \in P_2 \}$$

$$0\ [f(x) = ax^3 + bx^2 + cx + d]$$

$$1\ [f'(x) = 3ax^2 + 2bx + c]$$

$$2\ [f''(x) = 6ax + 2b]$$

$$= 3ax^2 + (2b + 12a)x + c + 4b = 0x^2 + 0x + 0$$

$$\Rightarrow \left.\begin{array}{l} 3a = 0 \Rightarrow a = 0 \\ 2b + 12a = 0 \Rightarrow b = 0 \wedge \\ c + 4b = 0 \Rightarrow c = 0 \end{array}\right\}$$

$$f(x) = ax^3 + bx^2 + cx + d = 0x^3 + 0x^2 + 0x + d = d.$$

$$\Rightarrow \ker(T) = \text{span}\{1\}.$$

$$\text{Rng}(T) = \{ g(x) = mx^2 + nx + q \in P_2 \mid \exists f(x) = ax^3 + bx^2 + cx + d \in P_3 \text{ s.t. } T(f(x)) = mx^2 + nx + q \}$$

where: $T(f(x)) = 2f''(x) + f'(x) = mx^2 + nx + q$

$$= 2[6ax + 2b] + 3ax^2 + 2bx + c = mx^2 + nx + q$$

$$= 3ax^2 + (12a + 2b)x + 4b + c = mx^2 + nx + q.$$

$$= a(3x^2 + 12x) + b(2x + 4) + c(1).$$

$$= \text{span}\{3x^2 + 12x,\ 2x + 4,\ 1\}.$$

Since $\text{Nullity}(T) = 1$. (b/c $\ker(T) = \text{span}\{1\}$),

& $\text{Nullity}(T) + \text{Rank}(T) = \dim(P_3) = 4$

$1 \quad + \text{Rank}(T) = 4$

$\Rightarrow \text{Rank}(T) = 3.$

$\Rightarrow \text{Rng}(T) = \text{span}\{3x^2 + 12x, 2x + 4, 1\}.$