Quiz #6 ***Math 260*** ***11/12/20*** ***Name:*** KEY

Show all your work clearly. No Work, No Credit.

1. ***Put your name on this quiz and scan your works into one single pdf file.***
2. ***Check if you have turned in every page (include this page)***
3. ***Your pdf file must be readable (without any shadow or dark spots).***
4. ***This quiz must be submitted by 7:10pm tonight. Late submission will be penalized 30% of your scores.***

1. Let $B=\{3x^2-x+2, x^2+x-3, 2x-5\}$ be a basis of P_2 find $\left[f(x)\right]_B$ where $f(x)=4x^2+6x-14$ (5pts)

$$[f(x)]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \text{ where } f(x) = 4x^2+16x-14 = \alpha_1(3x^2-x+2)+\alpha_2(x^2+x-3)+\alpha_3(2x-5)$$

$$= (3\alpha_1+\alpha_2)x^2 + (-\alpha_1+\alpha_2+2\alpha_3)x + 2\alpha_1-3\alpha_2-5\alpha_3$$

$$\Rightarrow \begin{cases} 3\alpha_1+\alpha_2 = 4 \\ -\alpha_1+\alpha_2+2\alpha_3 = 6 \\ 2\alpha_1-3\alpha_2-5\alpha_3 = -14 \end{cases} \Rightarrow \cancel{3} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 6 \\ 3 & 1 & 0 & 4 \\ 2 & -3 & -5 & -14 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -1 & 1 & 2 & 6 \\ 0 & 4 & 6 & 22 \\ 0 & -1 & -1 & -2 \end{array}\right] = -2\left[\begin{array}{ccc|c} -1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 11 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow \begin{cases} \alpha_3 = 7 \\ \alpha_2 = 2-7 = -5 \\ -\alpha_1 - 5 + 14 = 6. \\ \alpha_1 = 9-6 = 3. \end{cases}$$

$$\Rightarrow [f(x)]_\beta = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$$

2. Determine the change – of – base matrix $[P]_{B\to C}$ where (5pts)

$B=\{\underbrace{-4+x-6x^2}_{v_1},\ \underbrace{6+2x^2}_{v_2},\ \underbrace{-6-2x+4x^2}_{v_3}\}$ *and* $C=\{1-x+3x^2,\ 2\ ,3+x^2\}$.

$$[P]_{B\to C} = \begin{bmatrix} | & | & | \\ [v_1]_C & [v_2]_C & [v_3]_C \\ | & | & | \end{bmatrix}$$

where: $\alpha_1(1-x+3x^2)+2\alpha_2+\alpha_3(3+x^2)$

$= \alpha_1+2\alpha_2+(-\alpha_1)x+(3\alpha_1+\alpha_3)x^2.$

$$\Rightarrow \begin{cases} \alpha_1+2\alpha_2 = -4,\ 6,\ -6 \\ -\alpha_1 = 1,\ 0,\ -2 \\ 3\alpha_1+\alpha_3 = -6,\ 2,\ 4. \end{cases}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 0 & -4 & 6 & -6 \\ -1 & 0 & 0 & 1 & 0 & -2 \\ 3 & 0 & 1 & -6 & 2 & 4 \end{array}\right] =$$

$\alpha_1 = -1,\ 0,\ 2.$

$3(-1)+\alpha_3 = 6,\ 2,\ 4.$

$\alpha_3 = 9,\ 5,\ 7$

$-1+2\alpha_2 = -4,\ 6,\ -6$

$\alpha_2 = -\frac{3}{2},\ \frac{7}{2},\ -\frac{5}{2}$

$$[P]_{B\to C} = \begin{bmatrix} -1 & 0 & 2 \\ -\frac{3}{2} & \frac{7}{2} & -\frac{5}{2} \\ 9 & 5 & 7 \end{bmatrix}$$

3. Let $V = P_1$ with a map $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) g(x)\, dx$ (10 pts)

a) Prove that $\{V, \langle f, g \rangle\}$ is an inner product vector space.

1) $\langle f, f \rangle = \int_{-1}^{1} f(x) \cdot f(x)\, dx \geq 0$ & $\langle f, f \rangle = 0$ iff $f(x) = 0$

2) $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx = \int_{-1}^{1} g(x) \cdot f(x) dx = \langle g, f \rangle$

3) $\langle \alpha f, g \rangle = \int_{-1}^{1} \alpha f(x) g(x) dx = \alpha \int_{-1}^{1} f(x) \cdot g(x) dx = \alpha \langle f, g \rangle$

4) $\langle f, g+h \rangle = \int_{-1}^{1} f(x) (g(x) + h(x)) dx = \int_{-1}^{1} f(x) g(x) dx + \int_{-1}^{1} f(x) h(x) dx$

$= \langle f, g \rangle + \langle f, h \rangle.$

b) Let $f(x) = 2x - 3$ *and* $g(x) = x + 1$. Determine $proj_{g(x)} f(x)$

$$\widehat{\text{Proj}}_{g(x)} f(x) = \frac{\langle f, g \rangle}{\|g\|^2} \cdot g(x)$$

where $\langle f, g \rangle = \int_{-1}^{1} (2x-3)(x+1)\,dx = \int_{-1}^{1} (2x^2 - x - 3)\,dx = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 3 \Big|_{-1}^{1}$

$$= \frac{2}{3} - \frac{1}{2} - 3 - \left(-\frac{2}{3} - \frac{1}{2} - 3\right) = \frac{4}{3}$$

$$\|g\|^2 = \langle g, g \rangle = \int_{-1}^{1} (x+1)^2\,dx = \frac{(x+1)^3}{3} \Big|_{-1}^{1} = \frac{1}{3}[8] = 8/3.$$

$$\Rightarrow \widehat{\text{Proj}}_{g(x)} f(x) = \frac{4/3}{8/3} \cdot (x+1) = 2(x+1)$$

4. Let $T: V \mapsto W$ be a linear transformation. Prove that Ker(T) and Im(T) are subspace of V and of W respectively. (5 pts)

Lecture notes.

5. For the following linear transformation. Find basis of Ker(T) and basis of Im(T), then determine whether T is injective, surjective or bijective. (25 pts)

a) $T: R^3 \mapsto P_2 \text{ by } T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a+2b)x^2 + (c-3b)x + a + b + c$

$\ker(T) = \left\{ \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, T(\vec{v}) = 0 \right\} \Rightarrow (a+2b)x^2 + (c-3b)x + a+b+c = 0x^2 + 0x + 0.$

$\Rightarrow \begin{cases} a + 2b = 0 \\ c - 3b = 0 \\ a + b + c = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A \Rightarrow \det(A) = \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$

$= -3 - 1 - 2(-1) = -2 \neq 0.$

Since A is a non-singular $\Rightarrow$ it only has a trivial solution,

$\Rightarrow \ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow T$ is injective.

$\operatorname{Im}(T) = mx^2 + nx + q \in P_2$ such that $\exists\ \vec{v} \in \mathbb{R}^3$.

$T(\vec{v}) = (a+2b)x^2 + (c-3b)x + a + b + c = mx^2 + nx + q$

$= (x^2+1)a + (2x^2 - 3x + 1)b + (x+1)c$

$= \text{span}\{x^2+1,\ 2x^2-3x+1,\ x+1\}$ $\rightarrow$ Basis of Im(T) $\{x^2+1, 2x^2-3x+1, x+1\}$

Since Nullity$(T) = 0 \Rightarrow \text{Rank}(T) = 3$ $\Rightarrow T$ is surjective.

$\Rightarrow T$ is an isomorphism.

b) $T : P_2 \mapsto M_{2\times 2}(R)$ by $T\left(ax^2+bx+c\right)=\begin{bmatrix} a-b & a+b+c \\ c-2b & a-2b \end{bmatrix}$

$$\ker(T) = \left\{ f(x) = ax^2+bx+c \in P_2 \;\middle|\; T(f(x)) = \begin{bmatrix} a-b & a+b+c \\ c-2b & a-2b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$\Rightarrow \begin{cases} a-b=0 \\ a+b+c=0 \\ c-2b=0 \\ a-2b=0 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow a=b=c=0.$$

$$\Rightarrow \ker(T) = \{0x^2+0x+0\} \Rightarrow T \text{ is injective.}$$

$$\operatorname{Im}(T) = \left\{ \begin{bmatrix} m & n \\ 0 & p \end{bmatrix} \in M_2(\mathbb{R}) \;\middle|\; \exists f(x) \in P_2 ;\ T(f) = \begin{bmatrix} m & n \\ 0 & p \end{bmatrix} \right\}$$

$$\Rightarrow T(f(x)) = \begin{bmatrix} a-b & a+b+c \\ c-2b & a-2b \end{bmatrix} = \begin{bmatrix} m & n \\ 0 & p \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ -2 & -2 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_3} \right\}$$

Since $\text{Nullity}(T) = 0 \Rightarrow \text{Rank}(T) = \dim(\operatorname{Im}(T))$

$\Rightarrow \text{Nullity}(T) + \text{Rank}(T) = 3.$

$\Rightarrow \text{Rank}(T) = 3 \Rightarrow \{A_1, A_2, A_3\}$ is L.I.

c) $T: P_3 \mapsto P_3$ by $T(f(x)) = f''(x) - 6f(x)$

$$\text{Ker}(T) = \{f(x) = ax^3 + bx^2 + cx + d \in P_3 \mid T(f(x)) = 0x^3 + 0x^2 + 0x + 0\}$$

where $T(f) = f''(x) - 6f(x)$

$$\{-6\}\ f(x) = ax^3 + bx^2 + cx + d$$
$$\{0\}\ f'(x) = 3ax^2 + 2bx + c$$
$$\{1\}\ f''(x) = 6ax + 2b$$

$$= (-6a)x^3 + (-6b)x^2 + (-6c + 6a)x - 6d - 2b.$$

$$\left.\begin{array}{l} -6a = 0 \Rightarrow a = 0 \\ -6b = 0 \Rightarrow b = 0 \\ -6c + 6a = 0 \Rightarrow c = 0 \\ -6d - 2b = 0 \Rightarrow d = 0 \end{array}\right\} \Rightarrow \text{Ker}(T) = \{0\} \Rightarrow T \text{ is injective.}$$

$$\text{Im}(T) = \{g(x) = mx^3 + nx^2 + px + q \in P_3 \mid \exists f(x) \in P_3,\ T(f(x)) = g(x)\}$$

$$T(f(x)) = f''(x) - 6f(x)$$
$$= -6ax^3 - 6bx^2 + (-6c + 6a)x - 6d - 2b.$$
$$= a(-6x^3 + 6x) + b(-6x^2 - 2) + c(-6x)$$
$$= \text{span}\{-6x^3 + 6x,\ -6x^2 - 2,\ 6x\} \doteq \text{span}\{B\}$$

Nullity $(T) = 0 \Rightarrow \text{Rank}(T) = 3 \Rightarrow B$ is a basis of $\text{Im}(T)$.

$\Rightarrow T$ is also surjective.

$\Rightarrow T$ is an isomophism.

d) $T: R^3 \mapsto R^2$ by $T(\vec{v}) = A\vec{v}$ where $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \end{bmatrix}$

$\ker(T) = \text{Nullspace}(A) = \left\{ \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3 \mid A\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow -2 \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & -7 \end{bmatrix} \Rightarrow -7y - 7z = 0$

$y = -z = t, z = t.$

$\left. \begin{array}{l} x + 3y + z = 0 \\ x + 3(-t) + t = 0 \\ x = 2t. \end{array} \right\} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\Rightarrow \ker(T) = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \Rightarrow T$ is Not injective.

$\text{Im}(T) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \mid \exists \vec{v} \in \mathbb{R}^3 \ni T(\vec{v}) = \begin{bmatrix} a \\ b \end{bmatrix} \right\}$

$T(v) = A\vec{v} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y + z \\ 2x - y - 5z \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} a \\ b \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ -1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

$= \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\} \Rightarrow L.D.$

Since $\text{Nullity}(T) = 1 \Rightarrow \text{Rank}(T) = 2$

$\Rightarrow \text{Im}(T) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$:

T is surjective.