Quiz #7 **Math 260** **11/27/20** **Name:** Key

- ***This quiz is due by 4:45pm on Saturday 11/28/20. Late submission or by email will not be accepted.***
- ***If you are working as a group, then submit the names of all members in your group, and circle the name of the person whose papers will be graded.***
- ***Scan your work as ONE pdf file and submit it thru your Canvas account. Check to make sure your pdf file is readable and it's in the right direction (no upside down or sideway).***

1. Determine matrix of representation $[T]_B^C$ of the following linear transformation: (8 pts)

a) $T: P_3 \mapsto R^3$ *by* $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$ with $B = \{1, x, x^2, x^3\}$ *and* $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$[T(1)]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $[T(x)]_C = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$[T(x^2)] = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$, $[T(x^3)]_C = \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}_C = \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$

$\Rightarrow [T]_B^C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$

b) Let $V = span\{e^{3x}, xe^{3x}, x^2e^{3x}\}$, with basis $B = C = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ where $T: V \mapsto V$ by $T(y) = y'' - 2y' - 3y$

$$[T(e^{3x})] = (e^{3x})'' - 2(e^{3x})' - 3e^{3x} \Rightarrow [T(e^{3x})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= 9e^{3x} - 6e^{3x} - 3e^{3x} = 0$$

$$[T(xe^{3x})]_C = (xe^{3x})'' - 2(xe^{3x})' - 3(xe^{3x})$$

$$= e^{3x}(9x+6) - 2e^{3x}(3x+1) - 3xe^{3x}$$

$$= e^{3x}[9x + 6 - 6x - 2 - 3x],$$

$$= [4e^{3x}]_C = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(x^2e^{3x})]_C = (x^2e^{3x})'' - 2(x^2e^{3x})' - 3(x^2e^{3x})$$

$$= e^{3x}(9x^2 + 12x + 2) - 2e^{3x}(3x^2 + 2x) - 3x^2e^{3x}$$

$$= e^{3x}[9x^2 + 12x + 2 - 6x^2 - 4x - 3x^2]$$

$$= e^{3x}[8x - 2]_C = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$$

$$\Rightarrow [T]_B^C = \begin{bmatrix} 0 & 4 & -2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Determine all eigenvalues and corresponding eigenvectors of the following matrices: (20 pts)

a) $A = \begin{bmatrix} 7 & 3 \\ -6 & 1 \end{bmatrix}$

$$p(\lambda) = \lambda^2 - 8\lambda + 25 = \lambda^2 - 8\lambda + 16 + 9 = 0$$

$$\Rightarrow (\lambda - 4)^2 = -9 \Rightarrow \lambda = 4 \pm 3i.$$

$$\lambda = 4 + 3i \Rightarrow \begin{bmatrix} 3-3i & 3 \\ -6 & -3-3i \end{bmatrix} \Rightarrow (3-3i)x + 3y = 0$$

$$x = 1 \Rightarrow y = -1 + i.$$

$$\vec{V}_{4+3i} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} \quad \& \quad \vec{V}_{4-3i} = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

b) $A = \begin{bmatrix} -3 & -2 \\ 5 & -5 \end{bmatrix} \Rightarrow p(\lambda) = \lambda^2 + 8\lambda + 25 = \lambda^2 + 8\lambda + 16 + 9 = 0$

$$(\lambda + 4)^2 = -9 \Rightarrow \lambda = -4 \pm 3i$$

$$\lambda = -4 + 3i \Rightarrow \begin{bmatrix} 1+3i & -2 \\ -5 & -1-3i \end{bmatrix} \Rightarrow (1+3i)x - 2y = 0$$

$$x = 2 \Rightarrow y = 1 + 3i$$

$$\vec{V}_{-4+3i} = \begin{bmatrix} 2 \\ 1+3i \end{bmatrix} \quad \& \quad \vec{V}_{-4-3i} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}$$

c) $A = \begin{bmatrix} -4 & -1/4 \\ 85 & 3 \end{bmatrix}$; $p(\lambda) = \lambda^2 + \lambda + \left(-12 + \frac{85}{4}\right)$

$= \lambda^2 + \lambda + \frac{37}{4}$

$= \lambda^2 + \lambda + \frac{1}{4} + \frac{36}{4} = \left(\lambda + \frac{1}{2}\right)^2 + 9$

$p(\lambda) = \left(\lambda + \frac{1}{2}\right)^2 + 9 = 0$

$\lambda = -\frac{1}{2} \pm 3i$

$\lambda = -\frac{1}{2} + 3i \Rightarrow \begin{bmatrix} -4+\frac{1}{2}-3i & -\frac{1}{4} \\ 85 & 3+\frac{1}{2}-3i \end{bmatrix} = \begin{bmatrix} -\frac{7}{2}-3i & -\frac{1}{4} \\ 85 & \frac{7}{2}-3i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left(-\frac{7}{2} - 3i\right)x - \frac{1}{4}y = 0$ $\Rightarrow V_{-\frac{1}{2}-3i} = \begin{bmatrix} 1 \\ -14-12i \end{bmatrix}$

$x = 1$, $= 4\left(-\frac{7}{2} - 3i\right) = -14 - 12i$

$$\vec{V}_{-\frac{1}{2}+3i} = \begin{bmatrix} 1 \\ -14 + 12i \end{bmatrix}.$$

d) $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \Rightarrow p(\lambda) = \det \begin{bmatrix} 2-\lambda & 2 & -1 \\ 2 & 1-\lambda & -1 \\ 2 & 3 & -1-\lambda \end{bmatrix} = (2-\lambda)\left[(1-\lambda)(-1-\lambda)+3\right] - 2\left[-2-2\lambda+2\right] - \left[6-2+2\lambda\right]$

$$\Rightarrow p(\lambda) = (2-\lambda)\left[\lambda^2+2\right] + 2\lambda - 4$$

$$= (2-\lambda)(\lambda^2+2) - 2(2-\lambda)$$

$$= (2-\lambda)(\lambda^2+2-2) = 0 \Rightarrow \lambda = 2, 0, 0.$$

$$\lambda = 0 \Rightarrow \begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow y = 0 \Rightarrow 2x - z = 0$$

$$z = 2x.$$

$$\vec{V}_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \vec{V}_0 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$$\lambda = 2 \Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & -1 \\ 2 & 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 2 & 3 & -3 \\ 0 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow 2y - z = 0 \Rightarrow z = 2y \qquad 2x - y - 2y = 0$$

$$x = \tfrac{3}{2}y.$$

$$\vec{V}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2}y \\ y \\ 2y \end{bmatrix} = \frac{1}{2}y \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \Rightarrow \vec{V}_2 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}.$$

A is defective.

e) $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 3 & -3 \end{bmatrix} \Rightarrow p(\lambda) = \det \begin{bmatrix} -2-\lambda & 1 & 0 \\ 1 & -1-\lambda & -1 \\ 1 & 3 & -3-\lambda \end{bmatrix}$

$$p(\lambda) = (-2-\lambda)\left[(-1-\lambda)(-3-\lambda)+3\right] - \left[-3-\lambda+1\right]$$

$$= -(2+\lambda)\left[\lambda^2+4\lambda+6\right] + \left[\lambda+2\right]$$

$$= -(2+\lambda)\left[\lambda^2+4\lambda+6-1\right]$$

$$= -(2+\lambda)(\lambda^2+4\lambda+5)$$

$$= -(2+\lambda)(\lambda^2+4\lambda+4+1).$$

$$= -(2+\lambda)\left[(\lambda+2)^2+1\right] = 0$$

$$\Rightarrow \lambda = -2, \ -2 \pm i$$

$$\lambda = -2+i \Rightarrow \begin{bmatrix} -i & 1 & 0 \\ 1 & 1-i & -1 \\ 1 & 3 & -1-i \end{bmatrix} \begin{matrix} \leftarrow -ix+y=0 \quad \begin{cases} x=1 \\ y=i \end{cases} \\ \leftarrow 1+(1-i)(i)-z=0. \\ 1+i-i^2-z=0 \Rightarrow z=2+i \end{matrix}$$

$$\vec{V}_{-2+i} = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} \Rightarrow \vec{V}_{-2-i} = \begin{bmatrix} 1 \\ -i \\ 2-i \end{bmatrix}$$

$$\lambda = -2 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} \begin{matrix} \Leftrightarrow y=0 \\ \rightarrow x-z=0 \\ z=x \end{matrix} \Rightarrow \vec{V}_{-2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

3. Determine whether the following matrices are diagonallizable. Where possible, a matrix S such that $S^{-1}AS = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ (10 pts)

a) $A = \begin{bmatrix} 1 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$ $p(\lambda) = \lambda^2 - \frac{4}{3}\lambda + \frac{4}{9} = 0$

$$9\lambda^2 - 12\lambda + 4 = 0$$

$$(3\lambda - 2)(3\lambda - 2) = 0$$

$$\lambda = \frac{2}{3}, \frac{2}{3}.$$

$\lambda = \frac{2}{3} \Rightarrow \begin{bmatrix} 1-\frac{2}{3} & 1/3 \\ -1/3 & \frac{1}{3}-\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & -1/3 \end{bmatrix} \Rightarrow \frac{1}{3}x + \frac{1}{3}y = 0$

$\Rightarrow \vec{V}_{2/3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow$ A is defective.

$\Rightarrow$ A is not diagonalizable.

b) $A = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow p(\lambda) = \det \begin{bmatrix} 4-\lambda & 0 & 0 \\ 3 & -1-\lambda & -1 \\ 0 & 2 & 1-\lambda \end{bmatrix}.$

$$p(\lambda) = (4-\lambda)\left[(-1-\lambda)(1-\lambda)+2\right] = 0$$

$$= (4-\lambda)\left[\lambda^2+1\right] = 0 \Rightarrow \lambda = \pm i, 4$$

$$\lambda = i \Rightarrow \begin{bmatrix} 4-i & 0 & 0 \\ 3 & -1-i & -1 \\ 0 & 2 & 1-i \end{bmatrix} \begin{array}{l} \leftarrow x = 0 \\ \leftarrow (-1-i)y - z = 0 \Rightarrow \begin{cases} y=-1 \\ z = 1+i \end{cases} \\ \end{array}$$

$$\Rightarrow \vec{V}_i = \begin{bmatrix} 0 \\ 1 \\ 1+i \end{bmatrix} ; \quad \vec{V}_{-i} = \begin{bmatrix} 0 \\ 1 \\ 1-i \end{bmatrix}.$$

$$\lambda = 4 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 3 & -5 & -1 \\ 0 & 2 & -3 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow 2y - 3z = 0 \Rightarrow \begin{cases} z = 2 \\ y = 3 \end{cases} \end{array}$$

$$3x - 5(3) - 2 = 0$$

$$3x - 17 = 0 \qquad x = \frac{17}{3}$$

$$V_4 = \begin{bmatrix} 17/3 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ A is non-defective $\Rightarrow$ it's diagonalizable.

where $S = \begin{bmatrix} \vec{V}_i & \vec{V}_{-i} & \vec{V}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 17/3 \\ 1 & 1 & 2 \\ 1+i & 1-i & 3 \end{bmatrix};$

then $S^{-1}AS = \text{diag}(i, -i, 4) = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1/2 & 5 \\ 1/6 & 2/3 \end{bmatrix} \rightarrow p(\lambda) = \lambda^2 - \frac{7}{6}\lambda - \frac{1}{2} = 0$

$$6\lambda^2 - 7\lambda - 3 = 0$$

$$(3\lambda + 1)(2\lambda - 3) = 0 \Rightarrow \lambda = -\frac{1}{3}, \frac{3}{2}$$

$$\lambda = -\frac{1}{3} \Rightarrow \begin{bmatrix} \frac{1}{2}+\frac{1}{3} & 5 \\ \frac{1}{6} & \frac{2}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & 5 \\ \frac{1}{6} & 1 \end{bmatrix} \leftarrow \frac{5}{6}x + 5y = 0 \begin{cases} x = 6 \\ y = -1 \end{cases}, \quad \begin{bmatrix} 6 \\ -1 \end{bmatrix}.$$

$$\vec{V}_{-\frac{1}{3}} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\lambda = \frac{3}{2} \Rightarrow \begin{bmatrix} \frac{1}{2}-\frac{3}{2} & 5 \\ \frac{1}{6} & \frac{2}{3}-\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ \frac{1}{6} & -\frac{5}{6} \end{bmatrix} \leftarrow -x + 5y = 0 \begin{cases} x = 5 \\ y = 1 \end{cases} \quad \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{V}_{\frac{3}{2}} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow S = \begin{bmatrix} 6 & 5 \\ -1 & 1 \end{bmatrix}.$$

Then $S^{-1}AS = \text{diag}\left(-\frac{1}{3}, \frac{3}{2}\right) = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$

4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ (6 pts)

a) Determine all eigenvalues and eigenvectors of A.

b) Reduce A to row-echelon form and determine the eigenvalues and eigenvectors of the resulting matrix. Are these the same as the eigenvalues and eigenvectors of A?

a) $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow p(\lambda) = \lambda^2 + \lambda - 6 = 0$

$(\lambda+3)(\lambda-2) = 0 \Rightarrow \lambda = -3, 2.$

$\lambda = -3 \Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \leftarrow 4x + 2y = 0 \Rightarrow \begin{matrix} x = 1 \\ y = -2 \end{matrix} \quad \vec{V}_{-3} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 2 \Rightarrow \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \leftarrow -x + 2y = 0 \Rightarrow \begin{matrix} x = 2 \\ y = 1 \end{matrix} \quad \vec{V}_{2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

b) $-2 \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -6 \end{bmatrix}$

find Eigenvalues/Eigenvectors of REF of $A = \begin{bmatrix} 1 & 2 \\ 0 & -6 \end{bmatrix}$

$\Rightarrow p(\lambda) = \lambda^2 + 5\lambda - 6 = 0.$

$(\lambda + 6)(\lambda - 1) = 0 \Rightarrow \lambda = -6, 1.$

$\Rightarrow$ Different Eigenvalues / different Eigenvectors.

5. If $\vec{v_1}, \vec{v_2}$ *and* $\vec{v_3}$ are linearly independent eigenvectors of A corresponding to the eigenvalue λ, and c_1, c_2 *and* c_3 are scalars (not all zero), show that $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$ is also an eigenvector of A corresponding to the eigenvalue λ. (6 pts)

Prf: $\vec{V_1}, \vec{V_2}, \vec{V_3}$ are eigenvectors corresponding to λ.

$\Rightarrow \quad A\vec{v_1} = \lambda \vec{v_1}, \quad A\vec{v_2} = \lambda\vec{v_2}, \quad A\vec{v_3} = \lambda\vec{v_3}.$

$$V = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} \Rightarrow A\vec{v} = A\left(c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}\right)$$

$$= A\vec{v} = c_1 A\vec{v_1} + c_2 A\vec{v_2} + c_3 A\vec{v_3}$$

$$= c_1\lambda\vec{v_1} + c_2\lambda\vec{v_2} + c_3\lambda\vec{v_3}$$

$$= \lambda\left(c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}\right) = \lambda\vec{v}.$$

$\Rightarrow \vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$ is an eigenvector for A corresponding to eigenvalue λ.