Quiz #8 **Math 260** **12/3/20** **Name:** Key

Show all your work clearly. No Work, No Credit.

1. Prove / disprove if the following is a subspace. Find a basis if it's a vector space. (15 pts)

a) $S = \left\{ \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3 \,\middle|\, \int_0^2 \left(ax^2 + bx + c\right) dx = 0 \right\}$

Let. $\vec{V_1} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$; $\vec{V_2} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \in S \xrightarrow{\text{N.t.S.}} \vec{V_1} + \vec{V_2} = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix} \in S.$

Since $\vec{V_1} \in S \Rightarrow \int_0^2 (a_1x^2 + b_1x + c_1)\,dx = 0$

& $\vec{V_2} \in S \Rightarrow \int_0^2 (a_2x^2 + b_2x + c_2)\,dx = 0$

$$\int_0^2 \left[(a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2)\right] dx$$

$$= \int_0^2 (a_1x^2 + b_1x + c_1)\,dx + \int_0^2 (a_2x^2 + b_2x + c_2)\,dx = 0 + 0 = 0 \checkmark$$

$\Rightarrow \vec{V_1} + \vec{V_2} \in S.$ (1)

For any $\alpha \in \mathbb{R} \Rightarrow$ N.t.S. $\alpha\vec{V_1} \in S.$

$$\int_0^2 (\alpha a_1x^2 + \alpha b_1x + \alpha c_1)\,dx = \alpha \int_0^2 (a_1x^2 + b_1x + c_1)\,dx = \alpha \cdot 0 = 0 \checkmark$$

$\Rightarrow \alpha\vec{V_1} \in S$ (2)

from (1) & (2) $\Rightarrow$ S is a subspace of $\mathbb{R}^3$

Given any $f(x) = ax^2 + bx + c \in S \Rightarrow \int_0^2 (ax^2 + bx + c)\,dx$

$$\Rightarrow \frac{a}{5}x^3 + \frac{1}{2}bx^2 + cx \Big|_0^2 = \frac{8}{3}a + 2b + 2c = 0.$$

$$\Rightarrow c = -\frac{4}{3}a - b.$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in S \Rightarrow \begin{bmatrix} a \\ b \\ -\frac{4}{3}a - b \end{bmatrix} = \frac{1}{3}a \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$\Rightarrow \text{Basis of } S = \left\{ \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

b) $S=\{p(x)\in P_3 \mid p''(x)-3p'(x)=0\}$

$p_1(x), p_2(x)\in S \xrightarrow{\text{N.t.S.}} p_1(x)+p_2(x)\in S.$

$p_1(x)\in S \Rightarrow p_1''(x)-3p_1'(x)=0$
$p_2(x)\in S \Rightarrow p_2''(x)-3p_2'(x)=0$

$\left(p_1(x)+p_2(x)\right)''-3\left(p_1(x)+p_2(x)\right)'$

$=\underbrace{\left(p_1''(x)-3p_1'(x)\right)}_{0}+\underbrace{\left(p_2''(x)-3p_2'(x)\right)}_{0}$

$= 0 + 0 = 0$

$\Rightarrow p_1(x)+p_2(x)\in S$ (1)

for any $\alpha\in\mathbb{R} \Rightarrow (\alpha p_1)''-3(\alpha p_1)'=\alpha\left[p_1''-3p_1'\right]=\alpha\cdot 0=0$

$\Rightarrow \alpha p_1(x)\in S$ (2)

from (1) & (2) $\Rightarrow$ S is a subspace of P_3.

Let $f(x)=ax^3+bx^2+cx+d.$

$\Rightarrow (-3)\ f'(x)=3ax^2+2bx+c$

$(1)\ f''(x)=6ax+2b$

$(-9a)x^2+(-6b+6a)x-3c+2b=0$

$\Rightarrow -9a=0 \Rightarrow a=0$
$-6b+6a=0\Rightarrow b=0$
$-3c+2b=0\Rightarrow c=0$

$\Rightarrow f(x)=ax^3+bx^2+cx+d=0x^2+0x+0x+d(1)$

$=\operatorname{Span}\{1\}.$

$\Rightarrow$ Basis of $S=\{1\}.$

c) $S = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R) \middle| 3a - 2b = c + d \right\}$

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in S \Rightarrow 3a_1 - 2b_1 = c_1 + d_1$

$A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in S \Rightarrow 3a_2 - 2b_2 = c_2 + d_2$

$A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \Rightarrow 3(a_1 + a_2) - 2(b_1 + b_2) \stackrel{?}{=} (c_1 + c_2) + (d_1 + d_2)$

$$\underbrace{(3a_1 - 2b_1)}_{c_1 + d_1} + \underbrace{(3a_2 - 2b_2)}_{c_2 + d_2} = (c_1 + c_2) + (d_1 + d_2)$$

$\Rightarrow A_1 + A_2 \in S.$

Let $\alpha \in \mathbb{R}$

$\Rightarrow \alpha A_1 = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix} \Rightarrow \underbrace{3\alpha a_1 - 2\alpha b_1}_{\alpha(3a_1 - 2b_1)} \stackrel{?}{=} \alpha c_1 + \alpha d_1.$

$$\alpha \underbrace{(3a_1 - 2b_1)}_{=} $$

$$\alpha(c_1 + d_1) \quad //$$

$\Rightarrow \alpha A_1 \in S \Rightarrow$

S is a subspace of $M_2(\mathbb{R})$

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \Rightarrow 3a - 2b = c + d \Rightarrow d = 3a - 2b - c$

$\Rightarrow A = \begin{bmatrix} a & b \\ c & 3a - 2b - c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

$\Rightarrow S = \text{Span} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{A_3} \right\}$

You can verify $\{A_1, A_2, A_3\}$ is L.I.

$\Rightarrow$ Basis of $S = \{A_1, A_2, A_3\}$

2. Determine a basis of Ker(T), Rng(T) and then indicate if T is injective, surjective , bijective. (15pts)

a) $T : M_2(R) \mapsto P_2 \ by \ T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (2a - b - 2c)x^2 + (2c + b + d)x + 2a + d$

$\ker(T) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2 \;\middle|\; T(A) = 0x^2 + 0x + 0 \right\}$

$\Rightarrow T(A) = (2a-b-2c)x^2 + (2c+b+d)x + 2a+d = 0$

$\Rightarrow \begin{cases} 2a-b-2c = 0 \\ 2c+b+d = 0 \\ 2a+d = 0 \end{cases} \Rightarrow \begin{bmatrix} 2 & -1 & -2 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & -1 & -2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} b + 2c + d = 0 \Rightarrow b = -2c-d \\ -2a + 2c + d - 2c = 0 \\ \Rightarrow a = \frac{1}{2}d. \end{array}$

$\Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T) \Rightarrow A = \begin{bmatrix} \frac{1}{2}d & -2c-d \\ c & d \end{bmatrix} = c\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}d\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow \ker(T) = \text{Span}\left\{ \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \right\} \Rightarrow$ Not injective.

$\text{Im}(T) = \left\{ f(x) = mx^2 + nx + q \in P_2 \;\middle|\; \exists A \in M_2(\mathbb{R}) ;\ T(A) = f(x) \right\}$

$(2a-b-2c)x^2 + (2c+b+d)x + 2a+d = f(x)$

where $T(A) = (2a-b-2c)x^2 + (2c+b+d)x + 2a+d + c(-2x^2+2x) + d(x+1)$

$\Rightarrow f(x) = a(2x^2+2) + b(-x^2+x) + c(-2x^2+2x) + d(x+1)$

$\Rightarrow \text{Im}(T) = \text{Span}\{2x^2+2, -x^2+x, -2x^2+2x, x+1\}$

Since $\text{Nullity}(T) = 2$ ~~and~~ and $\text{Nullity}(T) + \text{Rank}(T) = 4$

$\Rightarrow \text{Rank}(T) = 2.$

$\Rightarrow$ Basis of $\text{Im}(T) = \{2x^2+2, -x^2+x\}$

Not surjective.

b) $T: P_2 \mapsto R^3$; by $T\left(ax^2+bx+c\right)=\begin{bmatrix} 3a-c \\ 2b+c \\ a+b+c \end{bmatrix}$

$$\ker(T) = \left\{ f(x) \in P_2 \;\middle|\; T(f(x)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in R^3 \right\}$$

$$T(ax^2+bx+c) = \begin{bmatrix} 3a-c \\ 2b+c \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} a = \frac{1}{3}c \\ b = \frac{1}{2}c \end{array}$$

$$\frac{1}{3}c + \frac{1}{2}c + c = 0 \Rightarrow c = 0 \Rightarrow a = b = 0$$

$$\Rightarrow \ker(T) = \{0x^2 + 0x + 0\} \Rightarrow T \text{ is injective. } (1)$$

$$\operatorname{Im}(T) = \left\{ \begin{bmatrix} m \\ n \\ p \end{bmatrix} \in R^3 \;\middle|\; \exists f(x) \in P_2,\; T(f(x)) = \begin{bmatrix} m \\ n \\ p \end{bmatrix} \right\}.$$

where $\begin{bmatrix} m \\ n \\ p \end{bmatrix} = T(f(x)) = \begin{bmatrix} 3a-c \\ 2b+c \\ a+b+c \end{bmatrix} = a\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$$\Rightarrow \operatorname{Im}(T) = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

since $\text{Nullity}(T) = 0$ & $\text{Nullity}(T) + \text{Rank}(T) = 3$,

$$\Rightarrow \text{Rank}(T) = 3 \Rightarrow$$

$$\text{Basis of } \operatorname{Im}(T) = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow T \text{ is surjective. } (2)$$

from (1) & (2) $\Rightarrow$ T is an isomorphism.

3. Let $\{P_2, \langle f, g\rangle\}$ be an inner product vector space where $\langle f(x), g(x)\rangle = \int_0^2 f(x)g(x)\,dx$. Given $f(x) = 2x-3$ *and* $g(x) = x^2+1$. Determine $proj_{f(x)} g(x)$ (10 pts)

$$\text{proj}_{f(x)}\, g(x) = \frac{\langle g(x), f(x)\rangle}{\|f(x)\|^2} \cdot f(x).$$

where $\langle g(x), f(x)\rangle = \int_0^2 (x^2+1)(2x-3)\,dx = \int_0^2 (2x^3-3x^2+2x-3)\,dx$

$$= \frac{1}{2}(2)^4 - (2)^3 + (2)^2 - 6 = -2.$$

$$\|f(x)\|^2 = \int_0^2 (2x-3)^2\,dx = \frac{(2x-3)^3}{6}\Big|_0^2 = \frac{1}{6} + \frac{27}{6} = \frac{28}{6} = \frac{14}{3},$$

$$\Rightarrow \text{proj}_{f(x)}\, g(x) = \frac{-2}{14/3}(2x-3) = -\frac{3}{7}(2x-3)$$

4. Determine whether the following matrices is diagonalizable, where possible, find a matrix S such that $S^{-1}AS = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ (10 pts)

a) $A = \begin{bmatrix} -7 & 4 \\ -4 & 1 \end{bmatrix} \Rightarrow p(\lambda) = \lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)^2 = 0 \Rightarrow \lambda = -3, -3$

$\lambda = -3 \Rightarrow \begin{bmatrix} -4 & 4 \\ -4 & 4 \end{bmatrix} \Rightarrow -4x + 4y = 0 \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$

$\vec{V}_{-3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow A$ is defective

$\Rightarrow A$ is not diagonalizable.

b) $A=\begin{bmatrix}1&1&-1\\1&1&1\\-1&1&1\end{bmatrix} \Rightarrow p(\lambda)=\det\begin{bmatrix}1-\lambda&1&-1\\1&1-\lambda&1\\-1&1&1-\lambda\end{bmatrix}$

$$p(\lambda)=(1-\lambda)\left[(1-\lambda)^2-1\right]-\left[1-\lambda+1\right]-\left[1+1-\lambda\right]$$

$$=(1-\lambda)\left(1-2\lambda+\lambda^2-1\right)-2+\lambda-2+\lambda$$

$$=\lambda(1-\lambda)(\lambda-2)+2(\lambda-2).$$

$$=(\lambda-2)\left[\lambda-\lambda^2+2\right].$$

$$=-(\lambda-2)(\lambda^2-\lambda-2)$$

$$=-(\lambda-2)(\lambda-2)(\lambda+1)=0 \Rightarrow \lambda=2,2,-1.$$

$$\lambda=2 \Rightarrow \begin{bmatrix}-1&1&-1\\1&-1&1\\-1&1&-1\end{bmatrix}=\begin{bmatrix}-1&1&-1\\0&0&0\\0&0&0\end{bmatrix} \Rightarrow \begin{array}{l}-x+y-z=0\\ y=x+z\end{array}$$

$$\vec{V}_2=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\x+z\\z\end{bmatrix}=x\begin{bmatrix}1\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\1\\1\end{bmatrix} \Rightarrow E_2=\mathrm{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

$$\lambda=-1 \Rightarrow \begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&1\\2&1&-1\\-1&1&2\end{bmatrix}=\begin{bmatrix}1&2&1\\0&-3&-3\\0&3&3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1&2&1\\0&1&1\\0&0&0\end{bmatrix} \begin{array}{l}\Rightarrow y+z=0 \Rightarrow y=-z\\ \Rightarrow x+2(-z)+z=0 \Rightarrow x=3z\end{array}$$

$$\vec{V}_{-1}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3z\\-z\\z\end{bmatrix}=z\begin{bmatrix}3\\-1\\1\end{bmatrix} \Rightarrow \vec{V}_{-1}=\begin{bmatrix}3\\-1\\1\end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \text{Then } S^{-1}AS = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$