Exam #1 ***Math 290*** ***Name:*** KEY

3/23/21

Show all your work clearly. No Work, No Credit.

1. Determine if the following function is a solution of a differential equation. (10 pts)

a) $\{-8\}\ y = C_1 e^{\frac{2x}{3}} + C_2 e^{-4x} + \cos(2x) + \sin(2x)$ *for* $3y'' + 10y' - 8y = -40\sin(2x)$

$\{10\}\ y' = \frac{2}{3}C_1 e^{\frac{2}{3}x} - 4C_2 e^{-4x} - 2\sin(2x) + 2\cos(2x)$

$\{3\}\ y'' = \frac{4}{9}C_1 e^{\frac{2}{3}x} + 16C_2 e^{-4x} - 4\cos(2x) - 4\sin(2x)$

$$C_1 e^{\frac{2}{3}x}\underbrace{\left[-8 + \frac{20}{3} + \frac{4}{3}\right]}_{=0} + C_2 e^{-4x}\underbrace{\left[-8 - 40 + 48\right]}_{=0} + \cos(2x)\underbrace{\left[-8 + 20 - 12\right]}_{=0}$$

$$+ \sin(2x)\left[-8 - 20 - 12\right] = -40\sin(2x)$$

Yes, it's a solution.

b) $y = C_1\sqrt{x} + \frac{C_2}{\sqrt{x}} - 9\sqrt[3]{x}$ *for* $4x^2 y'' + 4xy' - y = 5\sqrt[3]{x}$

$\{-1\}\ y = C_1 x^{1/2} + C_2 x^{-1/2} - 9x^{1/3}$

$\{4x\}\ y' = \frac{1}{2}C_1 x^{-1/2} - \frac{1}{2}C_2 x^{-3/2} - 3x^{-2/3}$

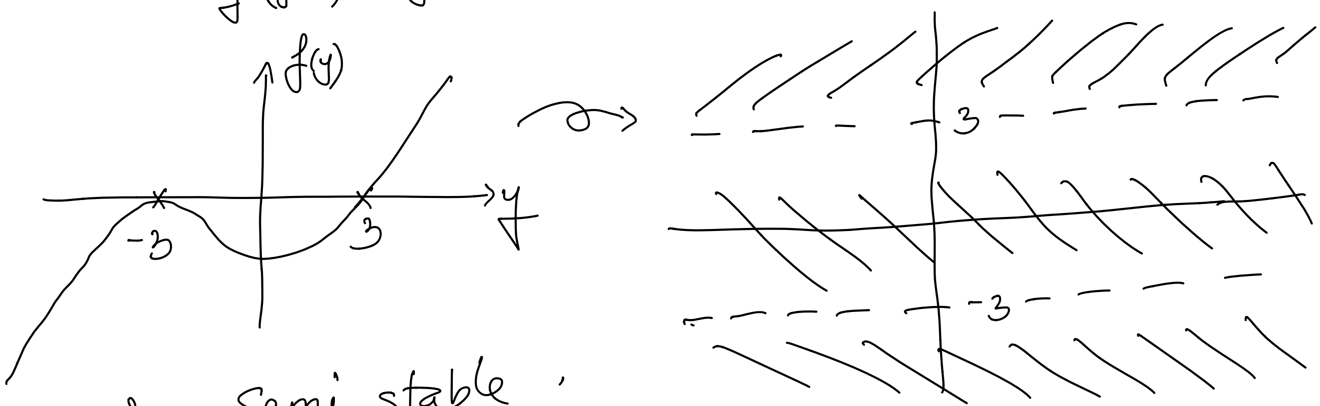
$\{4x^2\}\ y'' = -\frac{1}{4}C_1 x^{-3/2} + \frac{3}{4}C_2 x^{-5/2} + 2x^{-5/3}$

$$C_1\underbrace{\left[-x^{1/2} + 2x^{1/2} - x^{1/2}\right]}_{=0} + C_2\underbrace{\left[-x^{-1/2} - 2x^{-1/2} + 3x^{-1/2}\right]}_{=0} + \underbrace{\left[9x^{1/3} - 12x^{1/3} + 8x^{1/3}\right]}_{=5x^{1/3}} = 5\sqrt[3]{x}$$

Yes, it's a solution

2. Sketch the phase diagram and isocline of the following DE, then determine the stability of each equilibrium solutions. (10 pts)

a) $$\frac{dy}{dx} = \underbrace{y^3 + 3y^2}_{y^2(y+3)-9(y+3)\,=\,(y^2-9)(y+3)} - 9y - 27 = f(y) = (y-3)(y+3)^2.$$



$y = -3$: semi-stable,

$y = 3$: unstable.

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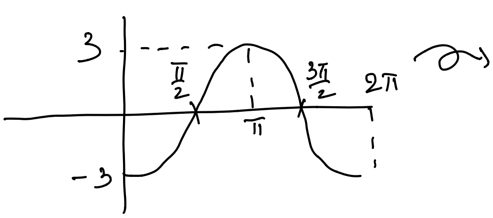
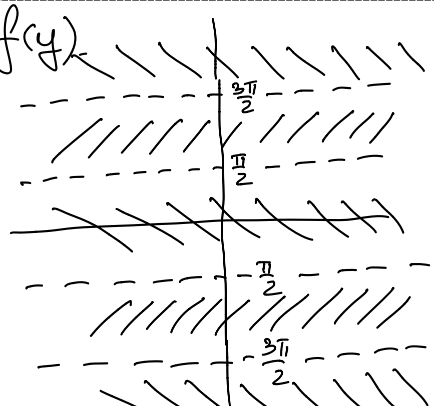
b) $$\frac{dy}{dx} = -3\cos(y) = f(y)$$





$y = \frac{\pi}{2} + 2n\pi$: unstable.

$y = \frac{3\pi}{2} + 2n\pi$: stable

4. Solve the following differential equations: (40 pts)

a) $\dfrac{dy}{dx} - y = f(x);\ y(0) = 3;\ f(x) = \begin{cases} x; & \text{if } 0 \le x < 2 \\ 0; & \text{if } x \ge 2 \end{cases}$ (8 pts)

$I(x) = e^{-\int dx} = e^{-x}$ $\Rightarrow$ $e^{-x}\dfrac{dy}{dx} - e^{-x}\cdot y = e^{-x} f(x)$

$\Rightarrow \dfrac{d}{dx}\left[e^{-x}\cdot y\right] = e^{-x} f(x)$

$\Rightarrow \int_0^x \dfrac{d}{dt}\left(e^{-t}\cdot y(t)\right)dt = \int_0^x e^{-t} f(t)\,dt.$

$e^{-t}\cdot y(t)\Big|_0^x = \int_0^x e^{-t} f(t)\,dt$

$e^{-x}\cdot y(x) - y(0) = \int_0^x e^{-t} f(t)\,dt \Rightarrow e^{-x} y(x) - 3 = \int_0^x e^{-t} f(t)\,dt.$

for $0 \le x < 2 \Rightarrow f(t) = t \Rightarrow e^{-x} y(x) - 3 = \int_0^x e^{-t}\cdot t\cdot dt$

t	e^{-t}
1	$-e^{-t}$
0	e^{-t}

$\Rightarrow e^{-x} y(x) - 3 = e^{-t}\left[-t-1\right]_0^x = e^{-x}(-x-1) - (-1) = -e^{-x}(x+1) + 1$

$e^{-x} y(x) = -e^{-x}(x+1) + 4 =$

$y(x) = -(x+1) + 4e^{x}.$

for $x \ge 2 \Rightarrow e^{-x} y(x) - 3 = \int_0^x e^{-t} f(t)\,dt = \int_0^2 e^{-t}\cdot t\,dt + \int_2^x e^{-t}\cdot 0\cdot dt$

$e^{-x} y(x) - 3 = e^{-t}(-x-1)\Big|_0^2$

$e^{-x} y(x) - 3 = e^{-2}(-3) - (-1) = 1 - 3e^{-2}$

$e^{-x} y(x) = 4 - 3e^{-2} \Rightarrow y(x) = 4e^{x} - 3e^{x-2}$

Solution: $y(x) = \begin{cases} 4e^{x} - (x+1) & \text{if } 0 \le x < 2. \\ 4e^{x} - 3e^{x-2} & \text{if } x \ge 2. \end{cases}$

b) $\underbrace{\left(ye^{xy}+\cos x\right)}_{M}dx+\underbrace{\left(xe^{xy}\right)}_{N}dy=0;\ y\left(\frac{\pi}{2}\right)=0$ (8pts)

$$\left.\begin{array}{l} M_y = e^{xy}+xye^{xy} \\ N_x = e^{xy}+xye^{xy} \end{array}\right\} \Rightarrow \text{Exact} \Rightarrow \text{Let } \Phi_y(x,y) = N = xe^{xy}$$

$$\Rightarrow \Phi(x,y) = \int xe^{xy}\,dy + h(x).$$

$$\Phi(x,y) = e^{xy} + h(x)$$

$$\Rightarrow \Phi_x(x,y) = ye^{xy} + h'(x) = M = ye^{xy} + \cos x.$$

$$\Rightarrow h'(x) = \cos x \Rightarrow h(x) = \int \cos x\,dx = \sin x$$

General $\text{sol}^{\underline{n}}$: $\Phi(x,y) = e^{xy} + \sin(x) = C.$

for $y\left(\frac{\pi}{2}\right)=0 \Rightarrow \Phi\left(\frac{\pi}{2},0\right) = e^{0} + \sin\frac{\pi}{2} = C \Rightarrow C = 2.$

$\underline{\text{Sol}^{n}}$: $\underline{\Phi(x,y) = e^{xy} + \sin x = 2}$

c) $x^2 \frac{dy}{dx} = y^2 + 3xy + x^2$ (8pts)

$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) + 1 \rightsquigarrow$ Let $V = \frac{y}{x} \Rightarrow y = xV \Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$

$V + x\frac{dV}{dx} = V^2 + 3V + 1 \Rightarrow x\frac{dV}{dx} = V^2 + 2V + 1 = (V+1)^2.$

$\Rightarrow \int \frac{dV}{(V+1)^2} = \int \frac{dx}{x} \Rightarrow \frac{(V+1)^{-1}}{-1} = \ln|x| + C.$

$\frac{1}{V+1} = -\ln|x| + C.$

$\frac{1}{\frac{y}{x} + 1} = -\ln|x| + C$

$\frac{x}{y+x} = C - \ln|x|.$

$\frac{x}{C - \ln|x|} = y + x$

General Sol$^{\underline{n}}$: $y = \frac{x}{C - \ln|x|} - x$

d) $\frac{dy}{dx}+\frac{2}{x}y=6\sqrt{1+x^2}\sqrt{y}$ (8pts)

$\frac{1}{y^{1/2}}\frac{dy}{dx}+\frac{2}{x}\cdot y^{1/2}=6\sqrt{1+x^2} \Rightarrow$ Let $V=y^{1-\frac{1}{2}}=y^{1/2} \Rightarrow \frac{dv}{dx}=\frac{1}{2}\cdot y^{-1/2}\cdot\frac{dy}{dx} \Rightarrow 2\frac{dv}{dx}=\frac{1}{y^{1/2}}\cdot\frac{dy}{dx}$

$\Rightarrow 2\frac{dV}{dx}+\frac{2}{x}V=6\sqrt{1+x^2} \Rightarrow \frac{dv}{dx}+\frac{1}{x}V=3\sqrt{1+x^2}$

$I(x)=e^{\int\frac{1}{x}dx}=e^{\ln x}=x \Rightarrow x\frac{dV}{dx}+V=3x\sqrt{1+x^2}$

$\int\frac{d}{dx}(x\cdot V)\,dx=3\int x\sqrt{1+x^2}\,dx$

Let $u=\sqrt{1+x^2}$
$u^2=1+x^2$
$2u\,du=2x\,dx$
$u\,du=x\,dx$

$xV=3\int u^2du=u^3+C$

$xy^{1/2}=(\sqrt{1+x^2})^3+C.$

$y^{1/2}=\frac{1}{x}\left[(1+x^2)^{3/2}+C\right].$

$y=\frac{1}{x^2}\left[(1+x^2)^{3/2}+C\right]^2.$

e) $\frac{dy}{dx} = (x+y+1)^2 - (x+y-1)^2$ (8 pts)

Let $v = x+y+1 \Rightarrow \frac{dv}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\Rightarrow x+y-1 = v-2$.

$\frac{dv}{dx} - 1 = v^2 - (v-2)^2 = v^2 - v^2 + 4v - 4$

$$\frac{dv}{dx} = 4v - 3 \Rightarrow \int \frac{dv}{4v-3} = \int dx$$

$$\frac{1}{4} \ln|4v-3| = x + C$$

$$\ln|4v-3| = 4x + c$$

$$4v - 3 = e^{4x+c} = ke^{4x}$$

$$4(x+y+1) = ke^{4x} + 3.$$

$$x+y+1 = ke^{4x} + \frac{3}{4}$$

$$y = ke^{4x} + \frac{3}{4} - x - 1 = ke^{4x} - x - \frac{1}{4}$$

6. A 500 – gallon - tank is partially filled with 200 gallons of fluid in which 3 pounds of salt are dissolved. Brine containing ½ pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well – mixed solution is then pumped out at a slower rate of 4 gal/min. (10 pts)

a) Let A(t) be the amount of salt in the tank at any time t (in mins), then set up an initial value problem.

b) Solve the differential equation of part (a).

c) Determine the concentration of salt in the tank when it's full.

a) $\frac{dA}{dt} = \text{rate in} - \text{rate out} = \frac{1}{2}(6) - \frac{A}{200+2t} \cdot 4 \quad ; A(0) = 3.$

$\Rightarrow \frac{dA}{dt} = 3 - \frac{2}{100+t} \cdot A \Rightarrow \frac{dA}{dt} + \frac{2}{100+t} A = 3$

$I(x) = e^{\int \frac{2}{100+t} dt} = e^{2\ln|100+t|} = (100+t)^2.$

$\int \frac{d}{dt}\left[(100+t)^2 A\right] dt = \int 3(100+t)^2 dt$

$(100+t)^2 A = (100+t)^3 + C \Rightarrow A(t) = 100 + t + C(100+t)^{-2}$

$A(0) = 100 + \frac{C}{(100)^2} = 3 \Rightarrow C = -97(100)^2.$

$A(t) = 100 + t - 97(100)^2(100+t)^{-2}$

time to fill $\Rightarrow$ Volume $200 + 2t = 600 \Rightarrow t = 200$ min.

$A(200) = 100 + 200 - 97(100)^2(100+200)^{-2}$

$= 300 - \frac{97(100)^2}{(300)^2} = 289.22$ lb.

Concentration: $\frac{289.22}{600} = 0.482$ lb/gal.

7. Solve the following differential equations: (30 pts)

a) $y''' - 4y'' + 5y' - 2y = 0$ (5 pts)

$P(\lambda) = \lambda^3 - 4\lambda + 5\lambda - 2 = 0$

$$\begin{array}{r|rrrr} \lambda = 1 & 1 & -4 & 5 & -2 \\ & & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1, 1, 2.$

General Sol: $y(x) = e^{x}(c_1 + c_2 x) + c_3 e^{2x}$

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b) $D^2 (D+2)^3 (4D^2 + 12D + 17) y = 0$ (5 pts)

$P(\lambda) = \lambda^2 (\lambda + 2)^3 (4\lambda^2 + 12\lambda + 17) = 0.$

$\lambda = 0, 0, -2, -2,$

$4\lambda^2 + 12\lambda + 17 = 0$

$\Rightarrow \underbrace{(2\lambda)^2 + 2(2\lambda)\cdot 3 + 9}_{} + 8 = 0$

$(2\lambda + 3)^2 + 8 = 0$

$\rightarrow 2\lambda + 3 = \pm\sqrt{-8} = \pm 2i\sqrt{2}$

$2\lambda = -3 \pm 2i\sqrt{2}$

$\lambda = -\frac{3}{2} \pm i\sqrt{2}$

General Soln:

$y(x) = c_1 + c_2 x + e^{-2x}(c_3 + c_4 x)$
$+ e^{-\frac{3}{2}x}\left(c_5 \cos(\sqrt{2}x) + c_6 \sin(\sqrt{2}x)\right).$

c) $4y'' - 4y' - 3y = 3x^2 - x - 2;\ y(0) = -4;\ y'(0) = 0$ (10 pts)

Homogeneous: $P(\lambda) = 4\lambda^2 - 4\lambda - 3 = 0$

$(2\lambda + 1)(2\lambda - 3) = 0 \Rightarrow \lambda = -\frac{1}{2}, \frac{3}{2}$

$y_h = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{3}{2}x}$

Particular sol^n: $\{-3\}\ y_p = Ax^2 + Bx + C$

$\{-4\}\ y_p' = 2Ax + B$

$\{4\}\ y_p'' = 2A$

$-3Ax^2 + (-3B - 8A)x - 3C - 4B + 8A = 3x^2 - x - 2.$

$\Rightarrow -3A = 3 \Rightarrow A = -1$

$-3B - 8A = -1 \Rightarrow -3B + 8 = -1 \Rightarrow B = 3.$

$-3C - 4B + 8A = -2 \Rightarrow -3C - 12 - 8 = -2 \Rightarrow C = -6$

General Sol: $y = y_h + y_p = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{3}{2}x} - x^2 + 3x - 6$

$y(0) = C_1 + C_2 - 6 = -4 \Rightarrow C_1 + C_2 = 2.$

$y'(0) = -\frac{1}{2}C_1 + \frac{3}{2}C_2 + 3 = 0 \Rightarrow -\frac{1}{2}C_1 + \frac{3}{2}C_2 = -3$

$\Rightarrow \begin{cases} C_1 + C_2 = 2 \\ -C_1 + 3C_2 = -6 \end{cases}$

$4C_2 = -4$

$C_2 = -1$

$C_1 = 3$

Solution:

$y(x) = -e^{-\frac{1}{2}x} + 3e^{\frac{3}{2}x} - x^2 + 3x - 6$

d) $9y''-12y'+13y=26x^2-35x+24$ (10 pts)

Homogeneous Sol$^{\underline{n}}$: $p(\lambda)=9\lambda^2-12\lambda+13=0$

$$\underbrace{(3\lambda)^2-2(3\lambda)\cdot 2+4}+9=0$$

$$(3\lambda+2)^2=-9 \Rightarrow 3\lambda+2=\pm 3i$$

$$\lambda=-\tfrac{3}{2}\pm i$$

$$y_h=e^{-\frac{3}{2}x}\left(C_1\cos x+C_2\sin x\right)$$

Particular Sol$^{\underline{n}}$: $\{13\}\ y_p=Ax^2+Bx+C$

$\{-12\}\ y_p'=2Ax+B$

$\{9\}\ y_p''=2A$

$$13Ax^2+(13B-24A)x+13C-12B+18A=26x^2-35x+24$$

$\Rightarrow 13A=26 \Rightarrow A=2$

$13B-24A=-35 \Rightarrow 13B-48=-35 \Rightarrow B=1$

$13C-12B+18A=24 \Rightarrow 13C-12+36=24 \Rightarrow C=0.$

General Sol$^{\underline{n}}$: $y=y_h+y_p=e^{-\frac{3}{2}x}\left(C_1\cos x+C_2\sin x\right)+2x^2+x.$