Quiz #1 **Math 290** **Name:** KEY

3/9/21

Show all your work clearly. No Work, No Credit.

1. Verify if the given function is a solution to the given differential equations: (12 pts)

a) $y = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x) + \frac{1}{9}e^{2x}$ *for* $y'' - 4y' + 13y = e^{2x}$

$y = e^{2x}\left[A\cos(3x) + B\sin(3x) + \frac{1}{9}\right]$ $\{13\}$

$y' = e^{2x}\left[2A\cos(3x) + 2B\sin(3x) + \frac{2}{9} - 3A\sin(3x) + 3B\cos(3x)\right]$ $\{-4\}$

$y'' = e^{2x}\left[4A\cos(3x) + 4B\sin(3x) + \frac{4}{9} - 6A\sin(3x) + 6B\cos(3x) - 6A\sin(3x) + 6B\cos(3x) - 9A\cos(3x) - 9B\sin(3x)\right]$ $\{1\}$

$$y'' - 4y' + 13y = e^{2x}\left\{\cos(3x)\left[13A - 8A - 12B + 4A + 6B + 6B - 9A\right] + \sin(3x)\left[13B - 8B - 12A + 4B - 6A - 6A - 9B\right] + \frac{13}{9} - \frac{8}{9} + \frac{4}{9}\right\} = e^{2x}$$

$\frac{13}{9} - \frac{8}{9} + \frac{4}{9} = \frac{9}{9} = 1 \Rightarrow$ Yes, it's a sol'n.

b) $\{6\}$ $y = Ax^2 + Bx^3 - x^2\sin x$ *for* $x^2y'' - 4xy' + 6y = x^4\sin x$

$\{-4x\}$ $y' = 2Ax + 3Bx^2 - 2x\sin x - x^2\cos x$.

$\{x^2\}\cdot$ $y'' = 2A + 6Bx - 2\sin x - 2x\cos x - 2x\cos x + x^2\sin x$

$$A\Big[\underbrace{6x^2 - 8x^2 + 2x^2}_{=0}\Big] + B\Big[\underbrace{6x^3 - 12x^3 + 6x^3}_{=0}\Big] + \sin x\Big[\underbrace{-6x^2 + 8x^2 - 2x^2 + x^4}_{=x^4}\Big] + \cos x\Big[\underbrace{4x^3 - 4x^3}_{=0}\Big]$$

$= x^4\sin x \Rightarrow y = Ax^2 + Bx^3 - x^2\sin x$ is a solution of the D.E.

c) $\{-35\}\cdot y = Ae^{7x} + Be^{-5x} + 3e^{2x}$ *for* $y'' - 2y' - 35y = e^{2x}$

$\{-2\}\cdot y' = 7Ae^{7x} - 5Be^{-5x} + 6e^{2x}$

$\{1\}\cdot y'' = 49Ae^{7x} + 25Be^{-5x} + 12e^{2x}$

$y'' - 2y' - 35y = Ae^{7x}(49 - 14 - 35) + Be^{-5x}(25 + 10 - 35) + e^{2x}(12 - 12 - 70)$

$= -70e^{2x} \neq e^{2x}.$

$\Rightarrow y = Ae^{7x} + Be^{-5x} + 3e^{2x}$ is Not a solution.

2. Determine all values of λ such that (6pts)

a) 12 $y = e^{\lambda x}$ is a solution of $2y''' - 3y'' - 8y' + 12y = 0$

$-8 \quad (y' = \lambda e^{\lambda x})$

$-3 \quad (y'' = \lambda^2 e^{\lambda x})$

$2 \quad (y''' = \lambda^3 e^{\lambda x})$

$2y''' - 3y'' - 8y' + 12y = e^{\lambda x}(2\lambda^3 - 3\lambda^2 - 8\lambda + 12) = 0$

$\underbrace{2\lambda^3 - 3\lambda^2}\underbrace{- 8\lambda + 12} = 0$

$\lambda^2(2\lambda - 3) - 4(2\lambda - 3) = 0$

$(\lambda^2 - 4)(2\lambda - 3) = 0$

$\lambda = \pm 2, \frac{3}{2}$

Sol^n: $y_1 = e^{2x}$, $y_2 = e^{-2x}$, $y_3 = e^{\frac{3}{2}x}$.

b) $\{-1\}\cdot y = x^{\lambda}$ is a solution of $x^2y'' + xy' - y = 0$

$\{x\}\cdot y' = \lambda x^{\lambda - 1}$

$\{x^2\}\cdot y'' = \lambda(\lambda - 1)x^{\lambda - 2}$

$x^2y'' + xy' - y = x^{\lambda}[\lambda^2 - \lambda + \lambda - 1] = 0.$

$= x^{\lambda}(\lambda^2 - 1) = 0. \Rightarrow \lambda = \pm 1$

sol: $y_1 = x$, $y_2 = x^{-1} = \frac{1}{x}.$

3. A 50 – gallon tank initially contains 20 gallons of salt water containing 8 pounds of salt. Suppose salt – water containing 0.5 lb/gal is pumped into the tank at the rate of 3 gal / min, which a well – mixed solution leaves the bottom of the tank at the rate of 2 gal / min. Find the concentration of salt in the tank when it's full. (7 pts)

Sol: let $A(t)$ be the amount of salt in the tank at time t (in mins.)

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dA}{dt} = 3\left(\frac{1}{2}\right) - \frac{A}{20+t}\cdot 2, \quad A(0)=8$$

$$\Rightarrow \frac{dA}{dt} + \frac{2}{20+t}A = 2.$$

$$I = e^{2\int \frac{1}{20+t}dt} = e^{2\ln|20+t|} = (20+t)^2.$$

$$\Rightarrow \int \frac{d}{dt}\left[(20+t)^2\cdot A\right]dt = \int 2(20+t)^2\,dt.$$

$$(20+t)^2\cdot A = \frac{2}{3}(20+t)^3 + C.$$

$$A(t) = \frac{2}{3}(20+t) + \frac{C}{(20+t)^2}$$

$$A(0) = \frac{40}{3} + \frac{C}{400} = 8 \Rightarrow C = 400\left[8 - \frac{40}{3}\right]$$

$$\Rightarrow C = -\frac{16(400)}{3} = -\frac{6400}{3}.$$

$$A(t) = \frac{2}{3}(20+t) - \frac{6400}{3(20+t)^2}.$$

full $\Rightarrow t = 30$ mins $\Rightarrow A(30) = \frac{100}{3} - \frac{6400}{3(2500)} = \frac{100}{3} - \frac{64}{75} = 32.48$ lbs.

$\Rightarrow$ Concentration: $\frac{32.48}{50} = 0.65$ lb/gal.

4. Solve the following DE. (25 pts)

a) $x\dfrac{dy}{dx} = y(\ln x - \ln y);\ y(1) = 4$

$x\frac{dy}{dx} = y \cdot \ln\left(\frac{x}{y}\right) \Rightarrow \frac{dy}{dx} = \frac{y}{x}\ln\left(\frac{x}{y}\right)$

let $v = \frac{y}{x} \Rightarrow y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = v\ln\left(\frac{1}{v}\right) = -v\ln v.$

$x\frac{dv}{dx} = -v\ln v - v.$

$\Rightarrow x\frac{dv}{dx} = -v(\ln v + 1) \Rightarrow \int \frac{dv}{v(\ln v+1)} = -\int \frac{1}{x}dx$

$\Rightarrow \ln|\ln v + 1| = -\ln|x| + C$

$\ln v + 1 = e^{-\ln x} \cdot e^{C} = \frac{K}{x}$

$\ln\left(\frac{y}{x}\right) + 1 = \frac{K}{x}$

$\ln y - \ln x + 1 = \frac{K}{x}$

$y(1) = 4 \Rightarrow \ln 4 - 0 + 1 = K \Rightarrow K = \underbrace{\ln 4 - 1}$
$= \ln\left(\frac{4}{e}\right)$

$\Rightarrow \underline{Sol}:\ \ln|y| - \ln x + 1 = \ln\left(\frac{4}{e}\right) \cdot \frac{1}{x}.$

b) $\dfrac{dy}{dx} = \dfrac{\sqrt[3]{2x-1}}{\sin(2x)} - 5\cot(2x)\,y$

$\frac{dy}{dx} + 5\cot(2x) \cdot y = \frac{\sqrt[3]{2x-1}}{\sin(2x)}$

$I = e^{5\int \cot(2x)\,dx} = e^{\frac{5}{2}\ln|\sin(2x)|} = (\sin(2x))^{5/2}$

$\Rightarrow \int \frac{d}{dx}\left[(\sin(2x))^{5/2} \cdot y\right] dx = \int \sqrt[3]{2x-1} \cdot (\sin(2x))^{3/2}\,dx$

$(\sin(2x))^{5/2} \cdot y = \int \sqrt[3]{2x-1}\,(\sin(2x))^{3/2}\,dx$

$y = \frac{1}{(\sin(2x))^{5/2}} \int \sqrt[3]{2x-1}\,(\sin(2x))^{3/2}\,dx.$

c) $\dfrac{dy}{dx} = \dfrac{y^2 + x\sqrt{x^2+y^2}}{xy} = f(x,y) = f(tx,ty) \leftarrow$ Homogeneous

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + \frac{\sqrt{x^2+y^2}}{y} = \left(\frac{y}{x}\right) + \sqrt{\left(\frac{x}{y}\right)^2 + 1}$$

$$\left\{ \begin{array}{l} \text{Let } v = y/x \\ y = xv \\ \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \end{array} \right.$$

$$v + x\frac{dv}{dx} = v + \sqrt{\frac{1}{v^2} + 1} \Rightarrow x\frac{dv}{dx} = \frac{\sqrt{1+v^2}}{v}.$$

$$\frac{1}{2}\int \frac{2v}{\sqrt{1+v^2}}\,dv = \int \frac{1}{x}\,dx \Rightarrow \frac{1}{2}\cdot 2\cdot\sqrt{1+v^2} = \ln|x| + C$$

$$\Rightarrow \sqrt{1+\left(\frac{y}{x}\right)^2} = \ln|x| + C \Rightarrow \sqrt{x^2+y^2} = x\left(\ln|x| + C\right).$$

d) $\dfrac{dy}{dx} = \sin(2x+2y-3) \rightarrow$ Let $v = 2x+2y-3 \Rightarrow \dfrac{dv}{dx} = 2 + 2\dfrac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{dv}{dx} - 1 \Rightarrow \frac{1}{2}\frac{dv}{dx} - 1 = \sin v.$$

$$\Rightarrow \frac{1}{2}\frac{dv}{dx} = 1 + \sin v \Rightarrow \int \frac{dv}{1+\sin v} = \int 2\,dx = 2x + C$$

$$\Rightarrow \int \frac{1-\sin v}{\cos^2 v}\,dv = 2x + C \Rightarrow \int \left(\sec^2 v - \sec v \tan v\right) dv = 2x + C$$

$$\Rightarrow \tan v - \sec v = 2x + C \Rightarrow \tan(2x+2y-3) - \sec(2x+2y-3) = 2x + C.$$

e) $$\underbrace{\left(e^x \sin y - 3x^2\right)}_{M} dx + \underbrace{\left(e^x \cos y + \frac{1}{3y^{2/3}}\right)}_{N} dy = 0$$

$$\left.\begin{array}{l} M_y = e^x \cos y \\ N_x = e^x \cos y \end{array}\right\} \Rightarrow \text{Exact} \Rightarrow \text{Let } \Phi_x = M \Rightarrow \Phi_x = e^x \sin y - 3x^2$$

$$\Rightarrow \Phi(x,y) = \int \left(e^x \sin y - 3x^2\right) dx + h(y)$$

$$\Phi(x,y) = e^x \sin y - x^3 + h(y)$$

$$\Rightarrow \Phi_y = e^x \cos y + h'(y) = N = e^x \cos y + \frac{1}{3y^{2/3}}$$

$$\Rightarrow h'(y) = \frac{1}{3} y^{-2/3} \Rightarrow h(y) = \frac{1}{3}\int y^{-2/3} dy = \frac{1}{3}\,\frac{3}{1}\,y^{1/3} = y^{1/3}$$

$$\Rightarrow \text{Solution: } \Phi(x,y) = \underline{e^x \sin y - x^3 + y^{1/3} = C}.$$