Quiz #2 *Math 290* *Name: KEY*

3/18/2021

Show all your work clearly. No Work, No Credit.

1. Solve the following differential equations: (25 pts

a) $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

$$e^x y \frac{dy}{dx} = e^{-y}\left(1 + e^{-2x}\right)$$

$$\Rightarrow y\frac{dy}{dx} = e^{-y}\left(\frac{1+e^{-2x}}{e^x}\right)$$

$$\Rightarrow \int y e^y\,dy = \int\left(e^{-x} + e^{-3x}\right)dx$$

$$\boxed{e^y(y-1) = -e^{-x} - \tfrac{1}{3}e^{-3x} + C}$$

b) $\dfrac{dy}{dx} + 2xy = f(x);\ f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases};\ y(0) = 2$

$$I(x) = e^{\int 2x\,dx} = e^{x^2} \Rightarrow \underbrace{e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y}_{''} = e^{x^2}f(x).$$

$$\frac{d}{dx}\left[e^{x^2}y\right] = e^{x^2}f(x)$$

$$\Rightarrow \int_0^x \frac{d}{dt}\left(e^{t^2}y(t)\right)dt = \int_0^x e^{t^2}f(t)\,dt$$

$$\Rightarrow e^{t^2}y(t)\Big|_0^x = \int_0^x e^{t^2}f(t)\,dt$$

$$e^{x^2}y(x) - y(0) = \int_0^x e^{t^2}f(t)\,dt$$

$$e^{x^2}y(x) - 2 = \int_0^x e^{t^2}f(t)\,dt$$

for $0 \le x < 1 \Rightarrow f(t) = t$

$$e^{x^2}y(x) - 2 = \int_0^x e^{t^2}t\,dt$$

$$e^{x^2}y(x) - 2 = \int_0^x e^{t^2}t\,dt \quad \begin{cases} u = t^2 \\ du = 2t\,dt \\ \tfrac{1}{2}du = t\,dt \end{cases}$$

$$e^{x^2}y(x) - 2 = \frac{1}{2}\int_0 e^u\,du = \frac{1}{2}e^{t^2}\Big|_0^x$$

$$= \frac{1}{2}\left[e^{x^2} - 1\right]$$

$$y(x) = e^{-x^2}\left[\frac{1}{2}e^{x^2} - \frac{1}{2} + 2\right]$$

$$y(x) = \frac{1}{2} + \frac{3}{2}e^{-x^2}$$

for $x > 1 \Rightarrow e^{x^2}y(x) - 2 = \int_0^x e^{t^2}f(t)\,dt = \int_0^1 e^{t^2}\cdot t\,dt + \int_1^x e^{t^2}\cdot 0\,dt$

$$\Rightarrow e^{x^2}y(x) - 2 = \frac{1}{2}e^{t^2}\Big|_0^1 = \frac{1}{2}\left[e - 1\right]$$

$$y(x) = e^{-x^2}\left[\frac{1}{2}e - \frac{1}{2} + 2\right] = e^{-x^2}\left[\frac{1}{2}e + \frac{3}{2}\right]$$

Sol^{n}: $y(x) = \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-x^2} & \text{if } 0 \le x < 1 \\ e^{-x^2}\left(\frac{1}{2}e + \frac{3}{2}\right) & \text{if } x \ge 1 \end{cases}$

c) $x^2\dfrac{dy}{dx}-2xy=3y^4,\ y(1)=\dfrac{1}{2}$ $\Rightarrow \dfrac{dy}{dx}-\dfrac{2}{x}y=\dfrac{3}{x^2}y^4$

$$\Rightarrow \underbrace{\frac{1}{y^4}\cdot\frac{dy}{dx}}-\frac{2}{x}\cdot\frac{1}{y^3}=\frac{3}{x^2} \Rightarrow \text{Let } V=y^{1-4}=y^{-3} \Rightarrow \frac{dV}{dx}=-3y^{-4}\cdot\frac{dy}{dx}$$

$$-\frac{1}{3}\frac{dV}{dx}-\frac{2}{x}V=\frac{3}{x^2} \Rightarrow \frac{dV}{dx}+\frac{6}{x}V=\frac{-9}{x^2}$$

$$\Rightarrow I=e^{\int\frac{6}{x}dx}=e^{6\ln|x|}=x^6 \Rightarrow \int\frac{d}{dx}\left[x^6\cdot V\right]dx=\int-9x^4\,dx$$

$$\Rightarrow x^6V=-\frac{9}{5}x^5+C \Rightarrow V=-\frac{9}{5x}+\frac{C}{x^6} \Rightarrow y^{-3}=\frac{9}{-5x}+\frac{C}{x^6}$$

$$y(1)=\frac{1}{2} \Rightarrow \frac{1}{8}=\frac{-9}{5}+C \Rightarrow C=\frac{1}{8}+\frac{9}{5}=\frac{77}{40}$$

$$\Rightarrow \text{sol}^n:\ \boxed{\frac{1}{y^3}=\frac{77}{40x^6}-\frac{9}{5x}}$$

d) $\dfrac{dy}{dx}=2+\sqrt{y-2x+3}$

$$\text{Let } V=y-2x+3 \Rightarrow \frac{dV}{dx}=\frac{dy}{dx}-2 \Rightarrow \frac{dy}{dx}=\frac{dV}{dx}+2$$

$$\Rightarrow \frac{dV}{dx}+2=2+\sqrt{V}. \Rightarrow \frac{dV}{dx}=\sqrt{V}.$$

$$\int\frac{dV}{\sqrt{V}}=\int dx \Rightarrow 2V^{1/2}=x+C$$

$$\Rightarrow 2\sqrt{y-2x+3}=x+C$$

e) $\left(x+ye^{y/x}\right)dx-xe^{y/x}dy=0,\ y(1)=0$

$$\left(x+ye^{y/x}\right)dx = xe^{y/x}dy$$

$$\frac{dy}{dx} = \frac{x+ye^{y/x}}{xe^{y/x}} = e^{-y/x}+\frac{y}{x}$$

Let $v=\frac{y}{x}$

$y = xv \Rightarrow \frac{dv}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = e^{-v} + v \Rightarrow x\frac{dv}{dx} = e^{-v}$$

$$\Rightarrow \int \frac{dv}{e^{-v}} = \int \frac{dx}{x}$$

$$\Rightarrow e^{v} = \ln|x| + c.$$

$$e^{y/x} = \ln|x| + c$$

$$y(1)=0 \Rightarrow e^{0} = \ln|1| + c \Rightarrow c = 1$$

$$\Rightarrow e^{y/x} = \ln|x| + 1.$$

$$\frac{y}{x} = \ln\left[\ln|x| + 1\right]$$

Sol: $y = x\ln\left[\ln|x|+1\right]$

2. Suppose there is a new kind of savings certificate that starts out paying 3% annual interest and increases the interest rate by 1% each additional year that the money is left on deposit. (Assume that interest is compounded continuously and that the interest rate increases continuously.) (10pts)

a) Write a DE for $\frac{dB}{dt}$, where $B(t)$ is the balance at time t.

b) Solve the equation that you found in part (a), assuming an initial deposit of $1000. And then find the balance of the account after 10 years.

$$\frac{dB}{dt} = (0.03 + 0.01t)\,B.$$

$$\Rightarrow \int \frac{dB}{B} = \int (0.03 + 0.01t)\,dt$$

$$\ln|B| = 0.03t + \frac{0.01}{2}t^2 + C.$$

$$B(t) = e^{0.03t + 0.005t^2} \cdot e^C = Ke^{0.03t + 0.005t^2}$$

b) $B(0) = \$1{,}000 \Rightarrow B(0) = K = 1{,}000$

$$\Rightarrow B(t) = 1000e^{0.03t + 0.005t^2}.$$

$$B(10) = 1000e^{0.03(10) + 0.005(10)^2} = \$2{,}225.54$$

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3. Solve the following DE: (15 pts)

a) $15y''-7y'-2y=0;\ y(0)=2;\ y'(0)=-1$

$P(\lambda)=15\lambda^2-7\lambda-2=0$

$(5\lambda+1)(3\lambda-2)=0 \Rightarrow \lambda=-\frac{1}{5},\frac{2}{3}$

$y(x)=C_1e^{-\frac{1}{5}x}+C_2e^{\frac{2}{3}x}.$

$y(0)=C_1+C_2=2$

$y'(0)=-\frac{1}{5}C_1+\frac{2}{3}C_2=-1$

$\Rightarrow \begin{cases} C_1+C_2=2 \\ -3C_1+10C_2=-15 \end{cases}$

$13C_2=-9$

$C_2=-\frac{9}{13} \Rightarrow C_1=2+\frac{9}{13}=\frac{35}{13}$

$\Rightarrow$ Solⁿ; $y(x)=\frac{35}{13}e^{-\frac{1}{5}x}-\frac{9}{13}e^{\frac{2}{3}x}.$

b) $2y''+5y'-3y=6x^2+x+17;\ y(0)=1;\ y'(0)=0$

Homogeneous $\Rightarrow P(\lambda)=2\lambda^2+5\lambda-3=(2\lambda-1)(\lambda+3)=0 \Rightarrow \lambda=\frac{1}{2},-3$

$\Rightarrow\ y_h=C_1e^{\frac{1}{2}x}+C_2e^{-3x}$

Consider a particular solution:

$-3\,[\,y_p=ax^2+bx+c\,]$

$5\,[\,y_p'=2ax+b\,]$

$2\,[\,y_p''=2a\,]$

$-3ax^2+(-3b+10a)x+(-3c+5b+4a)$

$=6x^2+x+17$

$\Rightarrow -3a=6 \Rightarrow a=-2.$

$-3b+10a=1 \Rightarrow b=-7$

$-3c+5b+4a=17 \Rightarrow c=-20$

$y_p=-2x^2-7x-20$

General sol: $y=y_h+y_p$

$y(x)=C_1e^{\frac{1}{2}x}+C_2e^{-x}-2x^2-7x-20.$

$y(0)=C_1+C_2-20=1 \Rightarrow C_1+C_2=21$

$y'(0)=\frac{1}{2}C_1-C_2-7=0 \qquad C_1-2C_2=14$

$3C_2=7$

$C_2=7/3$

$C_1=21-\frac{7}{3}$

$=\frac{56}{3}$

Sol: $y(x)=\frac{56}{3}e^{\frac{1}{2}x}+\frac{7}{3}-2x^2-7x-20$

c) $D^3(D+2)^2(4D^2+12D+17)y=0$

$$P(\lambda)=\lambda^3(\lambda+2)^2(4\lambda^2+12\lambda+17)=0$$

$$\lambda=0,0,0,-2,-2.$$

$$4\lambda^2+12\lambda+17=0$$

$$\underbrace{(2\lambda)^2+2(2\lambda)\cdot 3+9}+8=0$$

$$(2\lambda+3)^2=-8$$

$$2\lambda+3=\pm 2i\sqrt{2}.$$

$$\lambda=-\frac{3}{2}\pm i\sqrt{2}$$

$$y(x)=c_1+c_2x+c_3x^2+c_4e^{-2x}+c_5xe^{-2x}$$
$$+e^{-\frac{3}{2}x}\left(c_6\cos(\sqrt{2}x)+c_7\sin(\sqrt{2}x)\right).$$