Quiz #3 **Math 290** **Name: KEY**

4/7/21

Show all your work clearly. No Work, No Credit.

1. Solve the following: (5 pts / each = 40 pts)

a) $y''+\sqrt{2}y'-4y=0 \quad y(0)=0 \ \text{and} \ y'(0)=-1$

$$p(\lambda)=\lambda^2+\sqrt{2}\lambda-4=0 \Rightarrow \lambda=\frac{-\sqrt{2}\pm\sqrt{2+16}}{2}=\frac{-\sqrt{2}\pm3\sqrt{2}}{2}=\begin{cases}\frac{2\sqrt{2}}{2}=\sqrt{2}\\ -\frac{4\sqrt{2}}{2}=-2\sqrt{2}\end{cases}$$

$$y=c_1e^{\sqrt{2}x}+c_2e^{-2\sqrt{2}}$$

$$\left.\begin{aligned}&\left(y(0)=c_1+c_2=0\right)2\sqrt{2}\\ &y'(0)=\sqrt{2}c_1-2\sqrt{2}c_2=-1\\ \hline &3\sqrt{2}c_1=-1 \Rightarrow c_1=-\frac{1}{3\sqrt{2}}\\ &c_2=\frac{1}{3\sqrt{2}}\end{aligned}\right\} \quad y=-\frac{1}{3\sqrt{2}}e^{\sqrt{2}x}+\frac{1}{3\sqrt{2}}e^{-2\sqrt{2}x}.$$

b) $D^3(D-2)^2(4D^2+12D+17)y=0$

$$p(\lambda)=\lambda^3(\lambda-2)^2(4\lambda^2+12\lambda+17)=0$$

$$\lambda=0,0,0,2,2; \quad \Big[\underbrace{(2\lambda)^2+2(2\lambda)\cdot3+9}_{(2\lambda+3)^2=-8}+8\Big]=0$$

$$2\lambda+3=\pm2i\sqrt{2}.$$

$$\lambda=-\frac{3}{2}\pm i\sqrt{2}.$$

$$y=c_1+c_2x+c_3x^2+e^{2x}(c_4+c_5x)+e^{-\frac{3}{2}x}\left(c_6\cos(\sqrt{2}x)+c_7\sin(\sqrt{2}x)\right)$$

c) $y''-y=9xe^{2x};\ y(0)=0;\ y'(0)=7$

$P(\lambda)=\lambda^2-1=0 \Rightarrow \lambda=\pm1 \Rightarrow y_h=C_1e^{x}+C_2e^{-x}.$

$\{1\}\ y_p = e^{2x}(A+Bx)$

$\{0\}\ y_p' = e^{2x}(2A+2Bx+B)$

$\{-1\}\ y_p'' = e^{2x}(4A+4Bx+2B+2B)$

$= e^{2x}\left(-3A-4B+(B-4B)x\right) =$

$e^{2x}\left[-3A-3B-3Bx\right] = 9xe^{2x}$

$-3B=9 \Rightarrow B=-3$

$-3A-3B=0 \Rightarrow A=-B=3$

$y = C_1e^{x}+C_2e^{-x}+3-3x$

$y(0)=C_1+C_2+3=0$

$y'(0)=C_1-C_2-3=7$

$2C_1=7 \Rightarrow C_1=\frac{7}{2}$

$\frac{7}{2}+C_2+3=0 \Rightarrow C_2=-3-\frac{7}{2}=-\frac{13}{2}$

$y=\frac{7}{2}e^{x}-\frac{13}{2}e^{-x}+(3-3x)e^{2x}$

d) $6y''-y'-12y=32e^{-\frac{4}{3}x}-650\sin(x);\ y(0)=-2;\ y'(0)=3$

$P(\lambda)=6\lambda^2-\lambda-12=(3\lambda+4)(2\lambda-3)=0 \Rightarrow \lambda=-\frac{4}{3},\frac{3}{2}. \Rightarrow y_h=C_1e^{-\frac{4}{3}x}+C_2e^{\frac{3}{2}x}$

$\{-12\}\ y_{p_1}=Axe^{-\frac{4}{3}x}$

$\{-1\}\ y_{p_1}'=Ae^{-\frac{4}{3}x}\left[-\frac{4}{3}x+1\right]$

$\{6\}\ y_{p_1}''=Ae^{-\frac{4}{3}x}\left[\frac{16}{9}x-\frac{4}{3}-\frac{4}{3}\right]$

$Ae^{-\frac{4}{3}x}\left[-12x+\frac{4}{3}x+\frac{32}{3}x-1-16\right]=32e^{-\frac{4}{3}x}$

$\Rightarrow -17A=32 \Rightarrow A=-\frac{32}{17}$

$y_{p_1}=-\frac{32}{17}xe^{-\frac{4}{3}x}$

$\{12\}\ y_{p_2}=A\cos x+B\sin x$

$\{-1\}\ y_{p_2}'=-A\sin x+B\cos x$

$\{6\}\ y_{p_2}''=-A\cos x-B\sin x$

$\cos x\left[-12A-B-6A\right]+\sin x\left[-12B+A-6B\right]=-650\sin x$

$\begin{cases}-18A-B=0 \\ A-18B=-650\end{cases} \Rightarrow B=-18A \Rightarrow A-18(-18A)=-650 \Rightarrow 325A=-650 \Rightarrow \begin{cases}A=-2. \\ B=36.\end{cases}$

$y=C_1e^{-\frac{4}{3}x}+C_2e^{\frac{3}{2}x}-\frac{32}{17}xe^{-\frac{4}{3}x}-2\cos x+36\sin x$

$y(0)=C_1+C_2-2=-2 \Rightarrow C_1=-C_2$

$y'(0)=-\frac{4}{3}C_1+\frac{3}{2}C_2-\frac{32}{17}+36=3$

$\frac{4}{3}C_2+\frac{3}{2}C_2=-33+\frac{32}{17}=\frac{-529}{17}$

Solⁿ: $y=\frac{3174}{289}e^{-\frac{4}{3}x}-\frac{3174}{289}e^{\frac{3}{2}x}-\frac{32}{17}xe^{-\frac{4}{3}x}-2\cos x+36\sin x$

$$\frac{17}{6}c_2 = -\frac{529}{17} \Rightarrow c_2 = -\frac{529(6)}{(17)^2}$$

e) $2y''+5y'-3y = 8e^{3x}\cos(2x)$

$$P(\lambda) = 2\lambda^2+5\lambda-3 = (2\lambda-1)(\lambda+3)=0 \Rightarrow \lambda = \tfrac{1}{2}, -3 \Rightarrow y_h = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x}.$$

$$\{-3\}\ y_p = e^{3x}(A\cos(2x) + B\sin(2x))$$

$$\{5\}\ y_p' = e^{3x}\left[3A\cos(2x)+3B\sin(2x)-2A\sin(2x)+2B\cos(2x)\right]$$

$$\{2\}\ y_p'' = e^{3x}\left[\begin{array}{l}9A\cos(2x)+9B\sin(2x)-6A\sin(2x)+6B\cos(2x)-4A\cos(2x)-4B\sin(2x)\\ -6A\sin(2x)+6B\cos(2x)-4A\cos(2x)\end{array}\right]$$

$$e^{3x}\left\{\cos(2x)\left[-3A+15A+10B+18A+12B+12B-8A\right] + \sin(2x)\left[-3B+15B-10A+18B-12A-12A-8B\right]\right\}$$

$$\begin{array}{l} 34\ (22A+34B=8) \\ 22\ (-34A+22B=0) \end{array}$$

$$1640B = 272$$

$$B = \frac{34}{205};\ A = \frac{242}{2255}$$

$$\Rightarrow y(x) = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x} + e^{3x}\left[\frac{242}{2255}\cos(2x) + \frac{34}{205}\sin(2x)\right]$$

f) $2y''-5y'-3y = 21e^{-\frac{1}{2}x} + 5x^2 - 3x + 1$

$$P(\lambda) = 2\lambda^2-5\lambda-3 = (2\lambda+1)(\lambda-3)=0 \Rightarrow \lambda = -\tfrac{1}{2}, 3 \Rightarrow y_h = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$$

$$\{-3\}\ y_{p_1} = Axe^{-\frac{1}{2}x}$$

$$\{-5\}\ y_{p_1}' = Ae^{-\frac{1}{2}x}\left(-\tfrac{1}{2}x+1\right)$$

$$\{2\}\ y_{p_1}'' = Ae^{-\frac{1}{2}x}\left(\tfrac{1}{4}x-1\right)$$

$$Ae^{-\frac{1}{2}x}\left[-3x+\tfrac{5}{2}x+\tfrac{1}{2}x-5-2\right] = 21e^{-\frac{1}{2}x}$$

$$\Rightarrow -7A = 21 \Rightarrow A = -3.$$

$$\{-2\}\ y_{p_2} = Ax^2+Bx+C$$

$$\{-5\}\ y_{p_2}' = 2Ax+B$$

$$\{2\}\ y_{p_2}'' = 2A$$

$$-2Ax^2 + (-2B-10A)x - 2C-5B+4A = 5x^2-3x+1$$

$$\Rightarrow -2A = 5 \Rightarrow A = -\tfrac{5}{2};\quad -2B-10\left(-\tfrac{5}{2}\right) = -3 \Rightarrow B = 14;\ C = -\frac{81}{2}$$

$$y(x) = y_{p_2} = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x} - 3xe^{-\frac{1}{2}x} + \frac{5}{2}x^2 + 14x - \frac{81}{2}$$

6. Solve by elimination: (10 pts)

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2 + e^{2t} \\ \dfrac{dx_2}{dt} = 3x_1 - x_2 + 5e^{2t} \end{cases} \Rightarrow \begin{array}{l} (D+1) \\ \\ \end{array} \begin{cases} (D-1)x_1 - x_2 = e^{2t} \\ -3x_1 + (D+1)x_2 = 5e^{2t} \end{cases}$$

$$\left[D^2 - 1 - 3\right]x_1 = (D+1)e^{2t} + 5e^{2t}.$$

$$(D^2-4)x_1 = 2e^{2t} + e^{2t} + 5e^{2t} = 8e^{2t}.$$

$$P(\lambda) = \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2 \Rightarrow x_{1h} = c_1 e^{2t} + c_2 e^{-2t}.$$

$$\{-4\}\; x_{1p} = Ate^{2t}.$$

$$\{0\}\; x_{1p}' = Ae^{2t}[2t+1]$$

$$\{1\}\; x_{1p}'' = Ae^{2t}[4t+2+2]$$

$$Ae^{2t}[-4t+4t+4] = 8e^{2t}$$

$$4A = 8 \Rightarrow A = 2.$$

$$x_1 = c_1 e^{2t} + c_2 e^{-2t} + 2te^{2t}$$

$$(D-1)x_1 - x_2 = e^{2t} \Rightarrow x_2 = (D-1)x_1 - e^{2t}.$$

$$x_2(t) = (D-1)\left(c_1 e^{2t} + c_2 e^{-2t} + 2te^{2t}\right) - e^{2t}$$

$$= 2c_1 e^{2t} - 2c_2 e^{-2t} + 2e^{2t} + \underbrace{4te^{2t}} - c_1 e^{2t} - \underbrace{c_2 e^{-2t}} - \underbrace{2te^{2t}} - e^{2t}$$

$$x_2 = c_1 e^{2t} - 3c_2 e^{-2t} + 2te^{2t} + e^{2t}.$$

g) $4y''-4y'+y=e^{x/2}\sqrt{1-x^2}$

$P(\lambda)=4\lambda^2-4\lambda+1=(2\lambda-1)^2=0 \Rightarrow \lambda=\frac{1}{2},\frac{1}{2} \Rightarrow y=e^{\frac{1}{2}x}(c_1+c_2x)$ $\left\{\begin{array}{l} y_1=e^{\frac{1}{2}x} \\ y_2=xe^{\frac{1}{2}x}\end{array}\right.$

$$W[y_1,y_2]=\begin{vmatrix} e^{\frac{1}{2}x} & xe^{\frac{1}{2}x} \\ \frac{1}{2}e^{\frac{1}{2}x} & e^{\frac{1}{2}x}\left[\frac{1}{2}x+1\right]\end{vmatrix} = e^{x}\left[\frac{1}{2}x+1-\frac{1}{2}x\right]=e^{x}$$

$$u_1=-\int\frac{y_2\cdot F}{W[y_1,y_2]}dx=-\int\frac{xe^{\frac{1}{2}x}\cdot e^{\frac{1}{2}x}\sqrt{1-x^2}}{e^x}dx=\int x\sqrt{1-x^2}\,dx \quad \left\{\begin{array}{l} u=\sqrt{1-x^2} \\ u^2=1-x^2 \\ 2udu=-2xdx \\ -udu=xdx\end{array}\right.$$

$$\Rightarrow u_1=-\int u^2du=-\frac{1}{3}\left(\sqrt{1-x^2}\right)^3 ; \quad u_2=\int\frac{y_1\cdot F}{W[y_1,y_2]}dx=\int\frac{e^{\frac{1}{2}x}\cdot e^{\frac{1}{2}x}\cdot\sqrt{1-x^2}}{e^x}dx$$

$$u_2=\int\sqrt{1-x^2}\,dx \quad \left\{\begin{array}{l} \text{let } x=\sin\theta \\ dx=\cos\theta d\theta\end{array}\right. \Rightarrow \int\sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta=\int\cos^2\theta d\theta=\int\frac{1+\cos(2\theta)}{2}d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{2}\sin(2\theta)\right]=\frac{1}{2}\left[\theta+\sin\theta\sin\theta\right]=\frac{1}{2}\left[\sin^{-1}(x)+\frac{x}{1-x^2}\right]$$

Then the sol$^{\underline{n}}$: $y=e^{\frac{1}{2}x}\left[c_1+c_2x\right]+c_1u_1e^{\frac{1}{2}x}+c_2u_2xe^{\frac{1}{2}x}$.

h) $x^2y''+6xy'+6y=4e^{2x}$

Homogeneous $\left\{\begin{array}{l} \{6\}\ y=x^r \\ \{6x\}\ y'=rx^{r-1} \\ \{x^2\}\ y''=(r^2-r)x^{r-2}\end{array}\right\} \Rightarrow x^r\left[r^2-r+6r+6\right]=0 \Rightarrow r^2+5r+6=0$

$(r+2)(r+3)=0$

$r=-2,-3$

$y_h=c_1x^{-2}+c_2x^{-3}$ $\quad\left\{\begin{array}{l} y_1=x^{-2} \\ y_2=x^{-3}\end{array}\right. \Rightarrow W[y_1,y_2]=\begin{vmatrix} x^{-2} & x^{-3} \\ -2x^{-3} & -3x^{-4}\end{vmatrix}=-x^{-6}$

$$u_1=-\int\frac{y_2\cdot F}{W}=+\int\frac{x^{-3}\cdot 4e^{-2x}}{x^2\cdot x^{-6}}dx=4\int xe^{-2x}dx$$

x	e^{-2x}
1	$-\frac{1}{2}e^{-2x}$
0	$\frac{1}{4}e^{-2x}$

$$=4e^{-2x}\left(-\frac{1}{2}x-\frac{1}{4}\right)$$

$$u_2 = \int \frac{y_1 \cdot F}{W}\, dx = \int \frac{x^{-2} \cdot 4e^{2x}}{x^{2} \cdot (-x^{-6})}\, dx = -4\int x^{2} e^{2x}\, dx$$

x^2	$e^{2x}dx$
$2x$	$\frac{1}{2}e^{2x}$
2	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

$$= -4e^{2x}\left[\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right]$$

General Solution: $y = y_h + u_1 \cdot y_1 + u_2 \cdot y_2$

$$y = c_1 x^{-2} + c_2 x^{-3} + 4e^{2x}\left(-\frac{1}{2}x - \frac{1}{4}\right)x^{-2} - 4e^{2x}\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)$$