Quiz #4 **Math 290** **Name:** KEY.

4/15/21

Show all your work clearly. No Work, No Credit.

1. Solve the following: (20 pts)

a) $2y''+3y'-2y=20e^{-2x}+4x^2-2x+1$

Homogeneous: $p(\lambda)=2\lambda^2+3\lambda-2=0 \Rightarrow (2\lambda-1)(\lambda+2)=0 \Rightarrow \lambda=\frac{1}{2},-2$

$y_h=c_1e^{\frac{1}{2}x}+c_2e^{-2x}$

$\{-2\}\ y_{p_1}=Axe^{-2x}$

$\{3\}\ y'_{p_1}=Ae^{-2x}[-2x+1]$

$\{2\}\ y''_{p_1}=Ae^{-2x}[4x-2-2]$

$\Rightarrow Ae^{-2x}[8x-6x-2x-8+3]=20e^{-2x}$

$\Rightarrow A=-4 \quad \Rightarrow y_{p_1}=-4xe^{-2x}$

$\{-2\}\ y_{p_2}=ax^2+bx+c$

$\{3\}\ y'_{p}=2ax+b$

$\{2\}\ y''_{p_2}=2a$

$-2ax^2+(-2b+6a)x+(-2c+3b+4a)=4x^2-2x+1$

$\Rightarrow -2a=4 \Rightarrow a=-2,\quad -2b+6(-2)=-2 \Rightarrow b=-5$

$-2c+3(-5)+4(-2)=1 \Rightarrow c=\frac{1+8+15}{-2}=-6$

$y_{p_2}=-2x^2-5x-6$

sol^{n}: $y(x)=c_1e^{\frac{1}{2}x}+c_2e^{-2x}-4xe^{-2x}-2x^2-5x-6.$

b) $y''+6y'+13y=150\cos(2x);\ y(0)=1,\ y'(0)=3$

Homogeneous: $p(\lambda)=\lambda^2+6\lambda+13=\lambda^2+6\lambda+9+4=0 \Rightarrow (\lambda+3)^2=-4 \Rightarrow \lambda=-3\pm 2i$

$\Rightarrow y_h=e^{-3x}(c_1\cos(2x)+c_2\sin(2x))$

$\{13\}\ y_p=A\cos(2x)+B\sin(2x)$

$\{6\}\ y'_p=-2A\sin(2x)+2B\cos(2x)$

$\{1\}\ y''_p=-4A\cos(2x)-4B\sin(2x)$

$(13A+12B-4A)\cos(2x)+(13B-12A-4B)\sin(2x)=150\cos(2x)$

$4\ \{9A+12B=150$

$3\ \{-12A+9B=0$

$75B=150 \Rightarrow B=2.$

$12A=9B \Rightarrow A=\frac{3}{2}$

General Solution: $y(x)=e^{-3x}(c_1\cos(2x)+c_2\sin(2x))+\frac{3}{2}\cos(2x)+2\sin(2x)$

$y(0)=c_1+\frac{3}{2}=1 \Rightarrow c_1=1-\frac{3}{2}=-\frac{1}{2}$

$y'(x)=e^{-3x}(-3c_1\cos(2x)-3c_2\sin(2x)-2c_1\sin(2x)+2c_2\cos(2x))-3\sin(2x)+4\cos(2x)$

$y'(0)=-3\left(-\frac{1}{2}\right)+2c_2+4=3 \Rightarrow 2c_2=-1-\frac{3}{2}=-\frac{5}{2} \Rightarrow c_2=-\frac{5}{4}$

$\Rightarrow$ Sol: $y(x)=e^{-3x}\left[-\frac{1}{2}\cos(2x)-\frac{5}{4}\sin(2x)\right]+\frac{3}{2}\cos(2x)+2\sin(2x)$

c) $y''+4y'+4y=\dfrac{4e^{-2x}}{1+x^2}+2x^2-1$ $\Rightarrow p(\lambda)=\lambda^2+4\lambda+4=(\lambda+2)^2=0 \Rightarrow \lambda=-2,-2$ $\begin{cases} y_1=e^{-2x} \\ y_2=xe^{-2x}\end{cases}$

$\to y_h=(c_1+c_2x)e^{-2x}$

$W[y_1,y_2]=\det\begin{bmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}[-2x+1]\end{bmatrix} = e^{-4x}[-2x+1+2x]=e^{-4x}$

$y_{p_1}=u_1\cdot y_1+u_2\cdot y_2$; where $u_1=-\int\frac{y_2\cdot F}{W[y_1,y_2]}dx=-\int\frac{xe^{-2x}\cdot 4e^{-2x}}{(1+x^2)\cdot e^{-4x}}dx=4\int\frac{x}{1+x^2}dx$ $\begin{cases}\text{let } u=1+x^2\\ du=2x\,dx\\ \frac{du}{2}=x\,dx\end{cases}$

$u_1=2\int\frac{du}{u}=2\ln(1+x^2)$; $u_2=\int\frac{y_1\cdot F}{W[y_1,y_2]}dx=\int\frac{e^{-2x}\cdot 4e^{-2x}}{(1+x^2)\cdot e^{-4x}}dx=4\int\frac{1}{1+x^2}dx=4\tan^{-1}x$

$\left.\begin{aligned}\{4\}\ & y_{p_2}=ax^2+bx+c\\ \{4\}\ & y'_{p_2}=2ax\\ \{1\}\ & y''_{p_2}=2a\end{aligned}\right\}$ $4ax^2+(4b+8a)x+4c+2a=2x^2-1$ $\begin{cases}4a=2\Rightarrow a=2\\ 4b+8(2)=0\Rightarrow b=-4\\ 4c+2(2)=-1\Rightarrow c=-\frac{5}{4}\end{cases}$

Sol: $y(x)=(c_1+c_2x)e^{-2x}+2\ln(1+x^2)\cdot e^{-2x}+4\tan^{-1}x\cdot xe^{-2x}+2x^2-4x-\frac{5}{4}$

d) $x^2y''-3xy'+13y=0;\ y(1)=2,\ y'(1)=-5$

$\left.\begin{aligned} & y=x^r\\ & y'=rx^{r-1}\\ & y''=(r^2-r)x^{r-2}\end{aligned}\right.$ $\Rightarrow x^r\left[r^2-r-3r+13\right]=0\Rightarrow r^2-4r+13=0$

$\underbrace{r^2-4r+4}_{(r-2)^2}+9=0$

$(r-2)^2=-9\Rightarrow r=2\pm3i$

$\Rightarrow\ y(x)=x^2\left[c_1\cos(3\ln x)+c_2\sin(3\ln x)\right]$

$y(1)=c_1=2$.

$y'(x)=2x\left[c_1\cos(3\ln x)+c_2\sin(3\ln x)\right]+x^2\left[-c_1\sin(3\ln x)\cdot\frac{3}{x}+c_2\cos(3\ln x)\cdot\frac{3}{x}\right]$

$y'(1)=2[c_1]+3c_2=-5\Rightarrow 2(2)+3c_2=-5\Rightarrow c_2=-3$

Sol: $y(x)=x^2\left[2\cos(3\ln x)-3\sin(3\ln x)\right]$

2. Given $y''+4y'+7y=0;\ y(0)=-1;\ y'(0)=-1$ (10 pts)

a) Express the solution in phase – amplitude form.

b) If we think the equation as describing a mass – spring system, determine how often the mass crosses the equilibrium position.

c) Estimate the time t for which $|y(t)|<\dfrac{1}{100}$

a) $P(\lambda)=\lambda^2+4\lambda+7=\lambda^2+4\lambda+4+3=0 \Rightarrow (\lambda+2)^2=-3 \Rightarrow \lambda=-2\pm i\sqrt{3}$

$\Rightarrow y(t)=Ae^{-2t}\cos(\sqrt{3}t-\delta) \Rightarrow y(0)=A\cos(-\delta)=A\cos\delta=-1.$

$y'(t)=Ae^{-2t}\left[-2\cos(\sqrt{3}t-\delta)-\sqrt{3}\sin(\sqrt{3}t-\delta)\right]$

$y'(0)=A\left[-2\cos(-\delta)-\sqrt{3}\sin(-\delta)\right]=-2A\cos\delta+\sqrt{3}A\sin\delta=-1$

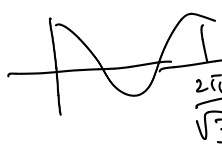
$\Rightarrow \quad 2+\sqrt{3}A\sin\delta=-1 \Rightarrow A\sin\delta=\frac{-3}{\sqrt{3}}=-\sqrt{3}.$

$\Rightarrow \left\{\begin{array}{l} A\cos\delta=-1 \Rightarrow A^2\cos^2\delta=1 \\ A\sin\delta=-\sqrt{3} \Rightarrow A^2\sin^2\delta=3 \end{array}\right\} \Rightarrow A^2=4 \Rightarrow A=2 \Rightarrow$

$\hookrightarrow \tan\delta=\sqrt{3} \Rightarrow \delta=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$ since $\left\{\begin{array}{l} A\cos\delta=-1 \\ A\sin\delta=-\sqrt{3} \end{array}\right\} \Rightarrow \delta\in Q\,III,$

$\Rightarrow \delta=\frac{\pi}{3}+\pi=\frac{4\pi}{3} \Rightarrow$ Sol: $y(x)=2e^{-2t}\cos\left(\sqrt{3}t-\frac{4\pi}{3}\right)$

b) Period: $\frac{2\pi}{\beta}=\frac{2\pi}{\sqrt{3}}$.



$\Rightarrow$ The mass will cross the equilibrium position: every $\frac{P}{4}=\frac{2\pi/\sqrt{3}}{4}=\frac{\pi}{2\sqrt{3}}$ sec.

c) for $|y(t)|<\frac{1}{100} \Rightarrow \left|2e^{-2t}\cos\left(\sqrt{3}t-\frac{4\pi}{3}\right)\right|<\frac{1}{100}.$

$\Rightarrow \quad 2e^{-2t}<0.01 \Rightarrow e^{-2t}<0.005$

$t>-\frac{1}{2}\ln(0.005)=2.65$ seconds.

3. A mass weight 32 lbs stretchs a spring 4ft beyond its natural length, the system is then placed in a medium that resists a damping force of three – quarter of instantaneous velocity. (10 pts)

a) Set up an IVP, then find a function described the motion of the mass.

b) The mass is released from ½ ft below the equilibrium position with an upward velocity of 2ft/sec. Determine speed of the object at t = 1second.

c) Estimate the time t for which $\left|y(t)\right| < \dfrac{1}{100}$

Sol: $W = mg \Rightarrow 32 = m\cdot(32) \Rightarrow m = 1\,\text{slug}$; $F = kx \Rightarrow 32 = k(4) \Rightarrow k = 8$

damping force: $c = \frac{3}{4}\cdot y'$.

$\Rightarrow$ a) IVP : $my'' + cy' + ky = 0 \Rightarrow y'' + \frac{3}{4}y' + 8y = 0$

$$\Rightarrow p(\lambda) = \lambda^2 + \frac{3}{4}\lambda + 8 = 0 \Rightarrow \underbrace{\lambda^2 + \frac{3}{4}\lambda + \frac{9}{64}}_{\left(\lambda + \frac{3}{8}\right)^2} = -8 + \frac{9}{64}$$

$$\left(\lambda + \frac{3}{8}\right)^2 = -\frac{503}{64} \Rightarrow \lambda = -\frac{3}{8} \pm \frac{i\sqrt{503}}{8}$$

b) $\Rightarrow$ $y(t) = Ae^{-\frac{3}{8}t}\cos\left(\frac{\sqrt{503}}{8}t - \delta\right)$ $\begin{cases} y(0) = \frac{1}{2} \\ y'(0) = -2 \end{cases}$

$$y(0) = A\cos(-\delta) = A\cos\delta = \frac{1}{2}$$

$$y'(t) = Ae^{-\frac{3}{8}t}\left[-\frac{3}{8}\cos\left(\frac{\sqrt{503}}{8}t - \delta\right) - \frac{\sqrt{503}}{8}\sin\left(\frac{\sqrt{503}}{8}t - \delta\right)\right]$$

$$y'(0) = A\left[-\frac{3}{8}\cos(-\delta) - \frac{\sqrt{503}}{8}\sin(-\delta)\right] = -2.$$

$$\Rightarrow \underbrace{-\frac{3}{8}A\cos\delta}_{-\frac{3}{16}} + \frac{\sqrt{503}}{8}A\sin\delta = -2.$$

$$-\frac{3}{16} + \frac{\sqrt{503}}{8}A\sin\delta = -2 \Rightarrow \frac{\sqrt{503}}{8}A\sin\delta = -2 + \frac{3}{16} = -\frac{29}{16}$$

$A\sin\delta = -\frac{29}{2\sqrt{503}} = -0.646 \Rightarrow A^2\sin^2\delta = 0.0042$; $\Rightarrow A^2 = 0.2542$

$A\cos\delta = \frac{1}{2} = 0.5 \Rightarrow A^2\cos^2\delta = 0.25$; $A = 0.5042$

$\hookrightarrow \tan\delta = -1.292 \Rightarrow \delta = \tan^{-1}(-1.292) = -0.912$. since $\delta \in QII$.

$\Rightarrow \delta = -0.912 + \pi = 2.23$

$\Rightarrow y(t) = 0.5042\, e^{-3/8t}\left(\cos\left(\frac{\sqrt{503}}{8}t - 2.23\right)\right) \Rightarrow$ speed at $t = 1$

$$y' = -0.5042e^{-\frac{3}{8}t}\left[-\frac{3}{8}\cos\left(\frac{\sqrt{503}}{8}t - 2.23\right) - \frac{\sqrt{503}}{8}\sin\left(\frac{\sqrt{503}}{8}t - 2.23\right)\right]$$

$$y'(1) = 0.5042e^{-\frac{3}{8}}\left[-\frac{3}{8}\cos\left(\frac{\sqrt{503}}{8} - 2.23\right) - \frac{\sqrt{503}}{8}\sin\left(\frac{\sqrt{503}}{8} - 2.23\right)\right] = -0.636\ \text{ft/sec}.$$

$$|y(t)| \le \left| 0.5042 e^{-\frac{3}{8}t} \right| \le \frac{1}{100} \Rightarrow e^{-\frac{3}{8}t} \le \frac{1}{50.42} \Rightarrow t \ge -\frac{8}{3}\ln\left(\frac{1}{50.42}\right) = 10.45 \text{ sec}.$$

4. Solve by elimination: (10 pts)

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 - x_2 \\ \dfrac{dx_2}{dt} = -x_1 + x_2 + 4e^t \end{cases} \Rightarrow \begin{array}{r} -1 \\ (D-2) \end{array} \begin{cases} (D-2)x_1 + x_2 = 0 \\ x_1 + (D-1)x_2 = 4e^t. \end{cases}$$

$$\left[-1 + (D-2)(D-1)\right]x_2 = (D-2)(4e^t)$$

$$\Rightarrow (D^2 - 3D + 1)x_2 = 4e^t - 8e^t = -4e^t.$$

$$P(\lambda) = \lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} \Rightarrow x_{1h} = C_1 e^{\frac{3+\sqrt{5}}{2}t} + C_2 e^{\frac{3-\sqrt{5}}{2}t}.$$

$$\left.\begin{array}{l} \{1\}\ x_{1p} = Ae^t. \\ \{-3\}\ x'_{1p} = Ae^t \\ \{1\}\ x''_{1p} = Ae^t \end{array}\right\} Ae^t(1-3+1) = -4e^t \Rightarrow A = 4 \Rightarrow x_{1p} = 4e^t.$$

$$x_1 = x_{1h} + x_{1p} = C_1 e^{\frac{3+\sqrt{5}}{2}t} + C_2 e^{\frac{3-\sqrt{5}}{2}t} + 4e^t.$$

$$x_2 = -(D-2)x_1 = -(D-2)\left[C_1 e^{\frac{3+\sqrt{5}}{2}t} + C_2 e^{\frac{3-\sqrt{5}}{2}t} + 4e^t\right]$$

$$= -\left[\frac{3+\sqrt{5}}{2}C_1 e^{\frac{3+\sqrt{5}}{2}t} + \frac{3-\sqrt{5}}{2}C_2 e^{\frac{3-\sqrt{5}}{2}t} + 4e^t - 2C_1 e^{\frac{3+\sqrt{5}}{2}t} - 2C_2 e^{\frac{3-\sqrt{5}}{2}t} - 8e^t\right]$$

$$= C_1 e^{\frac{3+\sqrt{5}}{2}t}\left(-\frac{3+\sqrt{5}}{2} + 2\right) + C_2 e^{\frac{3-\sqrt{5}}{2}t}\left(-\frac{3-\sqrt{5}}{2} + 2\right) - 4e^t$$

$$= \frac{1-\sqrt{5}}{2}C_1 e^{\frac{3+\sqrt{5}}{2}t} + \frac{1+\sqrt{5}}{2}C_2 e^{\frac{3-\sqrt{5}}{2}t} - 4e^t.$$

Sol:

$$\begin{cases} x_1 = c_1 e^{\frac{3+\sqrt{5}}{2}t} + c_2 e^{\frac{3-\sqrt{5}}{2}t} + 4e^t \\ x_2 = \frac{1-\sqrt{5}}{2} c_1 e^{\frac{3+\sqrt{5}}{2}t} + \frac{1+\sqrt{5}}{2} c_2 e^{\frac{3-\sqrt{5}}{2}t} - 4e^t \end{cases}$$