Quiz #5 **Math 290** **Name:** KEY

5/4/21

1. ***Put the names of all people in your group, and circle the name of the person whose papers will be graded.***
2. ***Scan your work as one pdf file and submit it thru Canvas.***

Show all your work clearly. No Work, No Credit.

1. For the following initial value problems model an idealized mass – spring system. (14pts)
   i) Find the solution.
   ii) Determine whether the mass ever crosses the equilibrium position and if so, determine the velocity of the mass at the instant it crosses the equilibrium.
   iii) Determine whether the mass ever crosses the equilibrium if the stated initial velocity is cut in half.

   a) $4x''+4x'+x=0;\ x(0)=2,\ x'(0)=-4$

i) $p(\lambda)=4\lambda^2+4\lambda+1=(2\lambda+1)^2=0 \Rightarrow \lambda=-\frac{1}{2},-\frac{1}{2}$

$x(t)=e^{-\frac{1}{2}t}(c_1+c_2t). \Rightarrow x'(t)=e^{-\frac{1}{2}t}\left[-\frac{1}{2}c_1-\frac{1}{2}c_2t+c_2\right]$

$x(0)=c_1=2$; $x'(0)=-\frac{1}{2}c_1+c_2=-4 \Rightarrow c_2=-3$

$x(t)=e^{-\frac{1}{2}t}(2-3t)$

ii) Cross the equilibrium $\Rightarrow x(t)=e^{-\frac{1}{2}t}(2-3t)=0 \Rightarrow t=\frac{2}{3}$

Velocity: $x'(t)=e^{-\frac{1}{2}t}\left(-1+\frac{3}{2}t-3\right)=e^{-\frac{1}{2}t}\left(\frac{3}{2}t-4\right)$

at $t=\frac{2}{3} \Rightarrow x'\left(\frac{2}{3}\right)=e^{-\frac{1}{2}\left(\frac{2}{3}\right)}\left(\frac{3}{2}\left(\frac{2}{3}\right)-4\right)=e^{-1}(-3)=-\frac{3}{e}$

iii) Half initial velocity $\Rightarrow x'(0)=\frac{1}{2}(-4)=-2$.

$x(t)=e^{-\frac{1}{2}t}(c_1+c_2t) \Rightarrow x'(t)=e^{-\frac{1}{2}t}\left[-\frac{1}{2}c_1-\frac{1}{2}c_2t+c_2\right]$

$x(0)=c_1=2$; $x'(0)=-\frac{1}{2}(2)+c_2=-2 \Rightarrow c_2=-1$

$\Rightarrow x(t)=e^{-\frac{1}{2}t}(2-t) \overset{\text{let}}{=\!=} 0 \Rightarrow t=2$ → Yes, it still crosses the equilibrium position.

b) $\frac{1}{2}x''+3x'+4x=0;\ x(0)=1,\ x'(0)=-3$

$P(\lambda)=\frac{1}{2}\lambda^2+3\lambda+4=0 \Rightarrow \lambda^2+6\lambda+8=(\lambda+2)(\lambda+4)=0 \Rightarrow \lambda=-2,-4$

i) $x(t)=c_1e^{-2t}+c_2e^{-4t} \Rightarrow x(0)=c_1+c_2=1$.

$x'(0)=-2c_1-4c_2=-3$

$2c_1=1 \Rightarrow c_1=\frac{1}{2}$.

$c_2=1-\frac{1}{2}=\frac{1}{2}$

$x(t)=\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-4t}$.

ii) Clearly it does not cross the equilibrium position.

iii) Half of initial velocity $\Rightarrow x'(0)=\frac{1}{2}(-3)=-\frac{3}{2}$

$\Rightarrow x(0)=c_1+c_2=1$.

$x'(0)=-2c_1-4c_2=-\frac{3}{2}$

$2c_1=4-\frac{3}{2}=\frac{5}{2}$

$c_1=\frac{5}{4}$

$c_2=1-\frac{5}{4}=-\frac{1}{4}$

$x(t)=\frac{5}{4}e^{-2t}-\frac{1}{4}e^{-4t}$

cross the equilibrium $\Rightarrow \frac{5}{4}e^{-2t}-\frac{1}{4}e^{-4t} \Rightarrow 5e^{-2t}=e^{-4t}$.

$\Rightarrow e^{2t}=\frac{1}{5} \Rightarrow t=\frac{1}{2}\ln\left(\frac{1}{5}\right)<0 \Rightarrow$ No it does not cross the equilibrium position.

2. Determine the term up to x^5 of the solution $y=\sum_{n=0}^{\infty} a_n x^n$ for $(x^2-3)y''-3xy'-5y=0$ (5 pts)

$$(-5)\ y=\sum_{n=0} a_n x^n$$
$$(-3x)\ y'=\sum_{n=1} a_n n x^{n-1}$$
$$(x^2-3)\ y''=\sum_{n=2} a_n n(n-1)x^{n-2}$$

$$\Rightarrow -5\sum_{n=0}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_n \cdot n\, x^n + \sum_{n=2}^{\infty} a_n \cdot n(n-1)x^n - 3\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}=0$$

$$-5\sum_{n=0}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n - 3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=0$$

Let $k=n-2 \Rightarrow n=k+2$

$$-5\sum_{n=0}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n - 3\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n=0$$

$$n=0 \Rightarrow -5a_0-3(2a_2)=0 \Rightarrow a_2=-\frac{5}{6}a_0$$

$$n=1 \Rightarrow -5a_1-3a_1-3(6)a_3=0 \Rightarrow a_3=-\frac{4}{9}a_1$$

$$\Rightarrow \sum_{n=2}^{\infty}\left[-5a_n-3na_n+n(n-1)a_n-3(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\Rightarrow (-5-3n+n^2-n)a_n-3(n+2)(n+1)a_{n+2}=0$$

$$\Rightarrow a_{n+2}=\frac{(n-5)(n+1)}{3(n+2)(n+1)}a_n=\frac{n-5}{3(n+2)}a_n$$

$n=0 \Rightarrow a_2=-\frac{5}{6}a_0$; $n=1 \Rightarrow a_3=-\frac{4}{9}a_1$.

$$n=2 \Rightarrow a_4=\frac{-3}{3(4)}a_2=-\frac{1}{4}a_2=-\frac{1}{4}\left(-\frac{5}{6}a_0\right)=\frac{5}{24}a_0.$$

$$n=3 \Rightarrow a_5=\frac{-2}{3(5)}a_3=-\frac{2}{15}\cdot\left(-\frac{4}{9}\right)a_1=\frac{8}{135}a_1$$

Sol:
$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$$
$$=a_0+a_1x-\frac{5}{6}a_0x^2-\frac{4}{9}a_1x^3+\frac{5}{24}a_0x^4+\frac{8}{135}a_1x^5+\cdots$$
$$=a_0\left[1-\frac{5}{6}x^2+\frac{5}{24}x^4+\cdots\right]+a_1\left[x-\frac{4}{9}x^3+\frac{8}{135}x^5-\cdots\right],$$

3. Determine the Laplace transform of the following functions: (5 pts)

a) $f(t) = 3t^3 + 2t - 3 + \sin(3t) + e^{-4t}$

$$\mathcal{L}(f(t)) = 3\mathcal{L}(t^3) + 2\mathcal{L}(t) - 3\mathcal{L}(1) + \mathcal{L}(\sin(3t)) + \mathcal{L}(e^{-4t})$$

$$= 3 \cdot \frac{3!}{s^4} + 2 \cdot \frac{1}{s^2} + 3 \cdot \frac{1}{s} + \frac{3}{s^2+9} + \frac{1}{s+4}$$

$$= \frac{18}{s^4} + \frac{2}{s^2} + \frac{3}{s} + \frac{3}{s^2+9} + \frac{1}{s+4}$$

b) $f(t) = e^{3t}(t^2+2)^2 - 5e^{-t}\sinh(4t)$

$$f(t) = e^{3t}(t^4 + 4t^2 + 4) - 5e^{-t}\sinh(4t).$$

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{3t} \cdot t^4) + 4\mathcal{L}(e^{3t} \cdot t^2) + 4\mathcal{L}(e^{3t}) - 5\mathcal{L}(e^{-t}\sinh(4t))$$

$$= \frac{4!}{(s-3)^5} + 4 \cdot \frac{2!}{(s-3)^3} + \frac{4}{s-3} - 5 \cdot \frac{4}{(s+1)^2 - 16}.$$

4. Determine inverse Laplace transform of the following functions: (6 pts)

a) $$F(s)=\frac{3s^2+22s+25}{(2s-1)(s+3)^2} = \frac{A}{2s-1}+\frac{B}{s+3}+\frac{C}{(s+3)^2}$$

$$3s^2+22s+25 = A(s+3)^2 + B(2s-1)(s+3)+C(2s-1)$$
$$=(A+2B)s^2+(6A+3B-B+2C)s+9A-3B-C$$

$A+B=3$

$6A+2B+2C=22$

$9A-3B-C=25$

where $C\big|_{s=-3} = \frac{27-66+25}{-7}=2$

$A\big|_{s=\frac{1}{2}} = \frac{\frac{3}{4}+11+25}{(\frac{1}{2}+3)^2} = \frac{147/4}{49/4} = 3 \Rightarrow B=0.$

$$\Rightarrow F(s)=\frac{3}{2s-1}+\frac{2}{(s+3)^2} \Rightarrow \mathcal{L}^{-1}(F(s)) = \frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{s-\frac{1}{2}}\right)+2\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2}\right)$$

$$=\frac{3}{2}e^{\frac{1}{2}t}+2e^{-3t}\cdot t.$$

b) $$F(s)=\frac{4s^2-9s-16}{s^3+6s^2+16s} = \frac{4s^2-9s-16}{s(s^2+6s+16)} = \frac{A}{s}+\frac{Bs+C}{s^2+6s+16}$$

$$4s^2-9s-16 = A(s^2+6s+16)+(Bs+C)s$$
$$=(A+B)s^2+(6A+C)s+16A$$

$A+B=4$

$6A+C=-9$

$16A=-16 \Rightarrow A=-1 \Rightarrow B=5,\ C=-3$

$$F(s)=\frac{-1}{s}+\frac{5s-3}{s^2+6s+16} = -\frac{1}{s}+\frac{5s-3}{(s^2+6s+9)+7}$$

$$=-\frac{1}{s}+\frac{5(s+3)-18}{(s+3)^2+7} = -\frac{1}{s}+\frac{5}{1}\cdot\frac{s+3}{(s+3)^2+7}-\frac{18}{\sqrt{7}}\cdot\frac{\sqrt{7}}{(s+3)^2+7}$$

$$\mathcal{L}^{-1}(F(s)) = -\mathcal{L}^{-1}\left(\frac{1}{s}\right)+5\mathcal{L}^{-1}\left(\frac{s+3}{(s+3)^2+7}\right)-\frac{18}{\sqrt{7}}\cdot\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{(s+3)^2+7}\right)$$

$$=-1+5e^{-3t}\cos(\sqrt{7}t)-\frac{18}{\sqrt{7}}e^{-3t}\sin(\sqrt{7}t)$$

5. Use the Laplace transform to solve the following IVP. (20 pts)

a) $y'' + y' - 2y = 10e^{-t};\ y(0) = 0,\ y'(0) = 1$

$\mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = 10\mathcal{L}(e^{-t})$

$s^2\mathcal{L}(y) - sy(0) - y'(0) - (s\mathcal{L}(y) - y(0)) - 2\mathcal{L}(y) = \frac{10}{s+1}$

$(s^2 + s - 2)\mathcal{L}(y) = \frac{10}{s+1} + 1 = \frac{10 + s + 1}{s+1} = \frac{s+11}{s+1}$

$\Rightarrow \mathcal{L}(y) = \frac{s+11}{(s+2)(s-1)(s+1)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{s+1}$

$A\big|_{s=-2} = \frac{9}{(-3)(-1)} = 3;\ B\big|_{s=1} = \frac{12}{(3)(2)} = 2,\ C\big|_{s=-1} = \frac{10}{-2} = -5$

$\mathcal{L}(y) = \frac{3}{s+2} + \frac{2}{s-1} - \frac{5}{s+1}$

$y(t) = 3\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - 5\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$

$y(t) = 3e^{-2t} + 2e^{t} - 5e^{-t}.$

b) $y''+3y'+2y=12te^{2t};\ y(0)=0;\ y'(0)=1$

$$\mathcal{L}(y'')+3\mathcal{L}(y')+2\mathcal{L}(y)=12\mathcal{L}(te^{2t})$$

$$s^2\mathcal{L}(y)-sy(0)-y'(0)+3\left[s\mathcal{L}(y)-y(0)\right]+2\mathcal{L}(y)=12\cdot\frac{1}{(s-2)^2}$$

$$(s^2+3s+2)\mathcal{L}(y)=\frac{12}{(s-2)^2}+1=\frac{12+s^2-4s+4}{(s-2)^2}$$

$$\mathcal{L}(y)=\frac{s^2-4s+16}{(s+1)(s+2)(s-2)^2}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s-2}+\frac{D}{(s-2)^2}$$

$$s^2-4s+16=A(s+2)(s-2)^2+B(s+1)(s-2)^2+C(s-2)(s+1)(s+2)+D(s+1)(s+2)$$

$$A\Big|_{s=-1}=\frac{1+4+16}{9}=\frac{21}{9}=\frac{7}{3}$$ and $A+B+C=0$

$$B\Big|_{s=-2}=\frac{4+8+16}{-16}=-\frac{7}{4}$$ $$C=-\frac{7}{3}+\frac{7}{4}=-\frac{7}{12}$$

$$D\Big|_{s=2}=\frac{4-8+16}{(3)(4)}=1$$

$$\Rightarrow \mathcal{L}(y)=\frac{7}{3}\cdot\frac{1}{s+1}-\frac{7}{4}\cdot\frac{1}{s+2}-\frac{7}{12}\cdot\frac{1}{s-2}+\frac{1}{(s-2)^2}$$

$$y(t)=\frac{7}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)-\frac{7}{4}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-\frac{7}{12}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)+\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$y(t)=\frac{7}{3}e^{-t}-\frac{7}{4}\mathcal{L}e^{-2t}-\frac{7}{12}e^{2t}+te^{2t}.$$