Quiz #6 **Math 290** **Name:** key

5/13/21

Show all your work clearly. No Work, No Credit.

1. Using power series $y=\sum_{n=0}^{\infty} a_n x^n$ to solve the DE: $y''-x^2y'-2y=0$ up to x^5. (5 pts)

$$-2\left[y=\sum_{n=0} a_n x^n\right]$$
$$(-x^2)\left[y'=\sum_{n=1} a_n \cdot n \cdot x^{n-1}\right]$$
$$1\left[y''=\sum_{n=2} a_n (n)(n-1)x^{n-2}\right]$$

$$\Rightarrow -2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n n x^{n+1} + \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} = 0$$

$(k=n)$ $(k=n+1)$ $n=k-1$ $(k=n-2)$ $(n=k+2)$

$$\Rightarrow -2\sum_{k=0}^{\infty} a_k x^k - \sum_{k=2}^{\infty} a_{k-1}(k-1)x^k + \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k = 0$$

$k=0 \Rightarrow -2a_0 + 2a_2 = 0 \Rightarrow a_2 = a_0$

$k=1 \Rightarrow -2a_1 x + 6a_3 x = 0 \Rightarrow a_3 = \frac{1}{3}a_1$

$$\Rightarrow \sum_{k=2}^{\infty}\left[-2a_k - a_{k-1}(k-1) + a_{k+2}(k+2)(k+1)\right]x^k = 0$$

$$\Rightarrow \text{for } k \geq 2 \Rightarrow -2a_k - a_{k-1}(k-1) + a_{k+2}(k+2)(k+1) = 0$$

$$\Rightarrow a_{k+2} = \frac{2a_k + (k-1)a_{k-1}}{(k+2)(k+1)}$$

k	a_k
0	$a_2 = a_0$
1	$a_3 = \frac{1}{3}a_1$
2	$a_4 = \frac{2a_2 + a_1}{(4)(3)} = \frac{1}{12}\left[2a_0 + a_1\right]$
3	$a_5 = \frac{2a_3 + 2a_2}{5(4)} = \frac{1}{10}(a_3 + a_2) = \frac{1}{10}\left(\frac{1}{3}a_1 + a_0\right)$

$$y = \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$
$$= a_0 + a_1 x + a_0 x^2 + \frac{1}{3}a_1 x^3 + \frac{1}{12}\left[2a_0 + a_1\right]x^4 + \frac{1}{10}\left(\frac{1}{3}a_1 + a_0\right)x^5 + \cdots$$
$$= a_0\left[1 + x^2 + \frac{1}{6}x^4 + \frac{1}{10}x^5 + \cdots\right] + a_1\left[x + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \cdots\right]$$

2. Determine the Laplace transform of the following functions: (10 pts)

a) $f(t) = e^{3t}\cos(3t) + (t^2 - 5t + 3)$

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{3t}\cos(3t)) + \mathcal{L}(t^2) - 5\mathcal{L}(t) + 3\mathcal{L}(1)$$

$$= \frac{s-3}{(s-3)^2+9} + \frac{2!}{s^3} - 5\left(\frac{1}{s^2}\right) + 3\left(\frac{1}{s}\right).$$

b) $f(t) = \begin{cases} 2 & if \ \ 0 \le t < 1 \\ 2e^{t-1} & if \ \ t > 1 \end{cases}$ $\Rightarrow f(t) = 2[1 - u_1(t)] + 2e^{t-1}u_1(t)$

$$f(t) = [2e^{t-1} - 2]u_1(t) + 2.$$

$$\Rightarrow \mathcal{L}(f(t)) = 2\mathcal{L}((e^{t-1} - 2)u_1(t)) + 2\mathcal{L}(1)$$

$$= 2e^{-s}\left[\frac{1}{s-1} - \frac{2}{s}\right] + \frac{2}{s}.$$

c) $f(t) = u_{\pi/2}(t)\cos(3t) - 3u_3(t)e^{2t+1}$

$$\mathcal{L}(f(t)) = \mathcal{L}(u_{\frac{\pi}{2}}(t)\cos(3t)) - 3\mathcal{L}(u_3(t)\cdot e^{2t+1})$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\left(\cos\left(3\left(t+\frac{\pi}{2}\right)\right)\right) - 3e^{-3s}\mathcal{L}\left(e^{2(t+3)+1}\right)$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\left[\cos(3t)\underbrace{\cos\left(\frac{3\pi}{2}\right)}_{=0} - \sin(3t)\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1}\right] - 3e^{-3s}\mathcal{L}(e^{2t}\cdot e^{7})$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}(\sin(3t)) - 3e^{7}\mathcal{L}(e^{2t})$$

$$= e^{-\frac{\pi}{2}s}\cdot\frac{3}{s^2+9} - 3e^{7}\cdot\frac{1}{s-2}.$$

3. Determine the inverse Laplace transform of the following: (10 pts)

a) $F(s)=\dfrac{48+s-11s^2}{(2s+3)(s^2+6s+14)} = \dfrac{A}{2s+3} + \dfrac{Bs+C}{s^2+6s+14}$

$$A\Big|_{s=-\frac{3}{2}} = \frac{48-\frac{3}{2}-\frac{99}{4}}{\frac{9}{4}-9+14} = 3$$

$$48+s-11s^2 = A(s^2+6s+14) + (2s+3)(Bs+C)$$

$$= (A+2B)s^2 + (6A+3B+2C)s + (14A+3C)$$

$$A+2B=-11 \Rightarrow 3+2B=-11 \Rightarrow B=-7 \;;\; 14A+3C=48$$

$$42+3C=48 \Rightarrow C=2.$$

$$\Rightarrow F(s) = \frac{3}{2s+3} + \frac{-7s+2}{s^2+6s+14} = \frac{3}{2}\cdot\frac{1}{s+\frac{3}{2}} + \frac{-7(s+3)+21+2}{(s^2+6s+9)+5}$$

$$\Rightarrow F(s) = \frac{3}{2}\cdot\frac{1}{s+\frac{3}{2}} - 7\cdot\frac{s+3}{(s+3)^2+5} + \frac{23}{\sqrt{5}}\cdot\frac{\sqrt{5}}{(s+3)^2+5}$$

$$\mathcal{L}^{-1}(F(s)) = \frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{s+\frac{3}{2}}\right) - 7\mathcal{L}^{-1}\left(\frac{s+3}{(s+3)^2+5}\right) + \frac{23}{\sqrt{5}}\mathcal{L}^{-1}\left(\frac{\sqrt{5}}{(s+3)^2+5}\right)$$

$$= \frac{3}{2}e^{-\frac{3}{2}t} - 7e^{-3t}\cos(\sqrt{5}t) + \frac{23}{\sqrt{5}}e^{-3t}\sin(\sqrt{5}t)$$

b) $F(s)=\dfrac{e^{-2s}(23-15s)}{(s+3)(s^2-4s+13)}$

$$A\Big|_{s=-3} = \frac{23+45}{9+12+13} = 2$$

$$F(s) = e^{-2s}\left[\frac{A}{s+3} + \frac{Bs+C}{s^2-4s+13}\right] \Rightarrow A(s^2-4s+13) + (Bs+C)(s+3) = 23-15s$$

$$(A+B)s^2 + (-4A+3B+C)s + 13A+3C = 23-15s$$

$$A+B=0 \Rightarrow B=-2 \;,\; 13A+3C=23 \Rightarrow 26+3C=23 \Rightarrow C=-1$$

$$F(s) = e^{-2s}\left[\frac{2}{s+3} + \frac{-2s-1}{s^2-4s+13}\right] = e^{-2s}\left[\frac{2}{s+3} - \frac{2(s-2)+4+1}{(s-2)^2+9}\right]$$

$$= e^{-2s}\left[\frac{2}{s+3} - 2\cdot\frac{s-2}{(s-2)^2+9} + \frac{5}{3}\cdot\frac{3}{(s-2)^2+9}\right]$$

$$\mathcal{L}^{-1}(F(s)) = y(t) = u_2(t)\left[2e^{-3(t-2)} - 2e^{2(t-2)}\cdot\cos\big(3(t-2)\big) + \frac{5}{3}e^{2(t-2)}\sin\big(3(t-2)\big)\right]$$

4. Express $L^{-1}(F(s)G(s))$ in terms of a convolution integral (Do not solve the integrals) (5 pts)

$$F(s)=\frac{2}{s^2+6s+10} \quad and \quad G(s)=\frac{2}{s-4}$$

$$F(s)=\frac{2}{s^2+6s+9+1}=\frac{2}{(s+3)^2+1};$$

$$\mathcal{L}^{-1}(F(s)\cdot G(s))=\mathcal{L}^{-1}\left[\frac{2}{(s+3)^2+1}\cdot\frac{2}{s-4}\right]=4\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2+1}\right)*\mathcal{L}^{-1}\left(\frac{1}{s-4}\right)$$

$$=4\cdot\left(e^{-3t}\sin t\right)*\left(e^{4t}\right)=4\int_0^t e^{-3(t-\tau)}\sin(t-\tau)\cdot e^{4\tau}\,d\tau.$$

5. Using Laplace transform to solve the following: (20 pts)

a) $y''-y=8\sin t-6\cos t;\ y(0)=2;\ y'(0)=-1$

$$\mathcal{L}(y'')-\mathcal{L}(y)=8\mathcal{L}(\sin t)-6\mathcal{L}(\cos t)$$

$$s^2\mathcal{L}(y)-2s+1-\mathcal{L}(y)=\frac{8}{s^2+1}-\frac{6s}{s^2+1}$$

$$(s^2-1)\mathcal{L}(y)=\frac{8-6s}{s^2+1}+2s-1=\frac{8-6s+2s^3+2s-s^2-1}{s^2+1}$$

$$\Rightarrow \mathcal{L}(y)=\frac{2s^3-s^2-4s+7}{(s-1)(s+1)(s^2+1)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}$$

$$2s^3-s^2-4s+7=A(s+1)(s^2+1)+B(s-1)(s^2+1)+(Cs+D)(s^2-1)$$

$$=(A+B+C)s^3+(A-B+D)s^2+(A+B-C)s+A+B-D$$

$A+B+C=2$

$A-B+D=-1$

$A+B-C=-4$

$A+B-D=7$

$2A=6$

$A=3$

$3+4+D=-1$

$D=-8$

$$\begin{cases}3+B+C=2\\3+B-C=-4\end{cases}$$

$6+2B=-2$

$B=-4,\ C=3$

$$\mathcal{L}(y)=\frac{3}{s-1}-\frac{4}{s+1}+\frac{3s-8}{s^2+1}$$

$$y=3\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)-4\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)+3\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)-8\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$\Rightarrow$ Sol^n: $y(t)=3e^{t}-4e^{-t}+3\cos t-8\sin t.$

b) $y''-4y=2te^t;\ y(0)=0;\ y'(0)=0$

$$\mathcal{L}(y'')-4\mathcal{L}(y)=2\mathcal{L}(te^t)$$

$$s^2\mathcal{L}(y)-4\mathcal{L}(y)=2\cdot\frac{1}{(s-1)^2}$$

$$(s^2-4)\mathcal{L}(y)=\frac{2}{(s-1)^2}$$

$$\mathcal{L}(y)=\frac{2}{(s-2)(s+2)(s-1)^2}$$

$$\mathcal{L}(y)=\frac{A}{s-2}+\frac{B}{s+2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$A\Big|_{s=2}=\frac{2}{4}=\frac{1}{2},\quad B\Big|_{s=-2}=\frac{2}{-4(-3)^2}=-\frac{1}{18}$$

$$D\Big|_{s=1}=\frac{2}{-3}=-\frac{2}{3}$$

$$s=0\Rightarrow\ \frac{2}{-4}=-\frac{1}{4}-\frac{1}{36}-C-\frac{2}{3}$$

$$-18=-9-1-36C-24$$

$$36C=-34+18=-16\Rightarrow C=-\frac{4}{9}$$

$$\mathcal{L}(y)=\frac{1}{2}\left(\frac{1}{s-2}\right)-\frac{1}{18}\left(\frac{1}{s+2}\right)-\frac{4}{9}\cdot\frac{1}{s-1}-\frac{2}{3}\cdot\frac{1}{(s-1)^2}$$

$$y(t)=\frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)-\frac{1}{18}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-\frac{4}{9}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)-\frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$$

Sol: $y(t)=\frac{1}{2}e^{2t}-\frac{1}{18}e^{-2t}-\frac{4}{9}e^{t}-\frac{2}{3}te^{t}$

c) $y'+2y=u_\pi(t)\sin(2t);\ y(0)=3$

$$\mathcal{L}(y')+2\mathcal{L}(y)=\mathcal{L}\left(u_\pi(t)\sin(2t)\right)$$

$$s\mathcal{L}(y)-3+2\mathcal{L}(y)=e^{-\pi s}\mathcal{L}\left(\sin(2(t+\pi))\right).$$

$$(s+2)\mathcal{L}(y)-3=e^{-\pi s}\mathcal{L}\left(\sin(2t)\cos(2\pi)+\cos(2t)\sin(2\pi)\right)$$

$$(s+2)\mathcal{L}(y)=e^{-\pi s}\mathcal{L}\left(\sin(2t)\right)+3.=e^{-\pi s}\cdot\frac{2}{s^2+4}+3.$$

$$\mathcal{L}(y)=e^{-\pi s}\cdot\frac{2}{(s+2)(s^2+4)}+\frac{3}{s+2}.$$

$$=e^{-\pi s}\left[\frac{A}{s+2}+\frac{Bs+C}{s^2+4}\right]+\frac{3}{s+2}.$$

$$\Rightarrow 2=A(s^2+4)+(Bs+C)(s+2)$$
$$=(A+B)s^2+(2B+C)s+4A+2C$$

$$\left\{\begin{array}{l} A\big|_{s=-2}=\frac{2}{8}=\frac{1}{4} \\ A+B=0\Rightarrow B=-\frac{1}{4}\end{array}\right. \quad \begin{array}{l} 4A+2C=2. \\ 1+2C=2 \\ C=\frac{1}{2}\end{array}$$

$$\mathcal{L}(y)=e^{-\pi s}\left[\frac{1}{4}\cdot\frac{1}{s+2}+\frac{-\frac{1}{4}s+\frac{1}{2}}{s^2+4}\right]+\frac{3}{s+2}.$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}\left[e^{-\pi s}\cdot\frac{1}{4}\cdot\frac{1}{s+2}-\frac{1}{4}\cdot\frac{s}{s^2+4}+\frac{1}{4}\cdot\frac{2}{s^2+4}\right]+3\cdot\frac{1}{s+2}$$

$$=u_\pi(t)\cdot\left[\frac{1}{4}e^{-2(t-\pi)}-\frac{1}{4}\cos\left(2(t-\pi)\right)+\frac{1}{4}\sin\left(2(t-\pi)\right)\right]+3e^{-2t}.$$