Quiz #7 **Math 290** ***6/1/21*** ***Name:*** KEY

Show all your work clearly. No Work, No Credit.

1. Solve the following DE: (10 pts)

1a) $\frac{dy}{dx}+\frac{2x}{1+x^2}y=xy^2;\ y(0)=1$

$\frac{1}{y^2}\frac{dy}{dx}+\frac{2x}{1+x^2}\cdot\frac{1}{y}=x \rightarrow$ Let $v=y^{1-2}=y^{-1} \Rightarrow \frac{dv}{dx}=-y^{-2}\cdot\frac{dy}{dx} \Rightarrow -\frac{dv}{dx}=\frac{1}{y^2}\cdot\frac{dy}{dx}$

$\underbrace{-\frac{dv}{dx}}+\frac{2x}{1+x^2}\cdot v=x \Rightarrow \frac{dv}{dx}-\frac{2x}{1+x^2}v=-x \Rightarrow$ Let $\mu=e^{-\int\frac{2x}{1+x^2}dx}=e^{-\ln(1+x^2)}=\frac{1}{1+x^2}$

$\Rightarrow \underbrace{\frac{1}{1+x^2}\cdot\frac{dv}{dx}-\frac{2x}{(1+x^2)^2}\cdot v}=\frac{-x}{1+x^2}$

$\int\frac{d}{dx}\left[\frac{1}{1+x^2}\cdot v\right]dx=-\int\frac{x}{1+x^2}dx$

$\frac{1}{1+x^2}\cdot v=-\frac{1}{2}\ln(1+x^2)+C.$

$\Rightarrow v=(1+x^2)\left[\ln\left(\frac{1}{\sqrt{1+x^2}}\right)+C\right]$

$\Rightarrow \frac{1}{y}=(1+x^2)\left[\ln\left(\frac{1}{\sqrt{1+x^2}}\right)+C\right]$

$y(0)=1 \Rightarrow 1=C \Rightarrow \frac{1}{y}=(1+x^2)\left[\ln\left(\frac{1}{\sqrt{1+x^2}}\right)+1\right]$

1b) $\underbrace{\left(ye^{xy}+\cos x\right)}_{M}dx+\underbrace{xe^{xy}}_{N}dy=0,\ y\left(\frac{\pi}{2}\right)=0$

$\left.\begin{array}{l}M_y=e^{xy}+xye^{xy}\\ N_x=e^{xy}+xye^{xy}\end{array}\right\}\Rightarrow$ Exact $\Rightarrow$ Let $\Phi_y=N=xe^{xy}$

$\Rightarrow \Phi=\int xe^{xy}dy+h(x)=e^{xy}+h(x).$

$\Rightarrow \Phi_x=ye^{xy}+h'(x)=M=ye^{xy}+\cos x \Rightarrow h'(x)=\cos x \Rightarrow h(x)=\sin x.$

Sol: $\Phi(x,y)=e^{xy}+\sin x=C.$

$y(\frac{\pi}{2})=0 \Rightarrow \Phi(\frac{\pi}{2},0)=e^0+\sin\frac{\pi}{2}=C \Rightarrow C=2.$

Ans: $\Phi(x,y)=\boxed{e^{xy}+\sin x=2.}$

2. Determine the general solution to the following DE: (10 pts)

2a) $y''+2y'+5y=34\cos(2x)$

Homogeneous: $p(\lambda)=\lambda^2+2\lambda+5=\underbrace{\lambda^2+2\lambda+1+4=0}_{(\lambda+1)^2=-4}\Rightarrow \lambda+1=\pm 2i$, $\lambda=-1\pm 2i$

$$y_h=e^{-x}\left[c_1\cos(2x)+c_2\sin(2x)\right]$$

particular sol:
$\{5\}\ y_p=A\cos(2x)+B\sin(2x)$
$\{2\}\ y_p'=-2A\sin(2x)+2B\cos(2x)$
$\{1\}\ y_p''=-4A\cos(2x)-4B\sin(2x)$

$\Rightarrow y''+2y'+5y=\left[5A+4B-4A\right]\cos(2x)+\left[5B-4A-4B\right]\sin(2x)=34\cos(2x)$
$=\left[A+4B\right]\cos(2x)+\left[-4A+B\right]\sin(2x)$

$$4\,(A+4B=34)$$
$$-4A+B=0$$
$$17B=136$$
$$B=\frac{136}{17}=8$$
$$A=34-4(8)=2.$$

General Sol$^{\underline{n}}$:
$$y=e^{-x}\left[c_1\cos(2x)+c_2\sin(2x)\right]+8\cos(2x)+2\sin(2x)$$

2b) $y''-2y'+y=\dfrac{e^x}{x^2+1}$

Homogeneous: $p(\lambda)=\lambda^2-2\lambda+1=(\lambda-1)^2=0\Rightarrow\lambda=1,1$

$y_h=e^x\left[c_1+c_2x\right]$ $\left\{\begin{array}{l} y_1=e^x\\ y_2=xe^x\end{array}\right.\Rightarrow W[y_1,y_2]=\det\begin{bmatrix} e^x & xe^x\\ e^x & e^x(x+1)\end{bmatrix}=e^{2x}(x+1)-xe^{2x}=e^{2x}$

$y_p=u_1\cdot y_1+u_2y_2$; where $u_1=-\displaystyle\int\frac{y_2\cdot F}{W[y_1,y_2]}dx=-\int\frac{xe^x\cdot e^x}{(x^2+1)\cdot e^{2x}}dx=-\int\frac{x}{x^2+1}dx=-\frac{1}{2}\ln(x^2+1)$

$$u_2=\int\frac{y_1F}{W[y_1,y_2]}dx=\int\frac{e^x\cdot e^x}{(x^2+1)\cdot e^{2x}}dx=\int\frac{1}{x^2+1}dx=\tan^{-1}x$$

General Sol$^{\underline{n}}$:

$$y=e^x\left[c_1+c_2x\right]-\frac{1}{2}\ln(x^2+1)\cdot e^x+\tan^{-1}x\cdot x\cdot e^x.$$

3. Determine the motion of the spring – mass system governed by the given initial – value problem. State whether the motion is underdamped, critically damped or over damped. Then determine the first time it passes the equilibrium position: $y''+4y'+7y=0, y(0)=2; \frac{dy}{dt}(0)=6$ (5pts)

$$p(\lambda)=\lambda^2+4\lambda+7=\lambda^2+4\lambda+4+3=0 \Rightarrow (\lambda+2)^2=-3 \Rightarrow \lambda=-2\pm i\sqrt{3}.$$

$\Rightarrow$ Under-dam.:

$$y(t)=Ae^{-2t}\cos(\sqrt{3}t-\delta).$$

$$y(0)=A\cos(-\delta)=A\cos\delta=2.$$

$$y'(t)=Ae^{-2t}\left[-2\cos(\sqrt{3}t-\delta)-\sin(\sqrt{3}t-\delta)\right]$$

$$y'(0)=A\left[-2\cos(-\delta)-\sin(-\delta)\right]=-2A\cos\delta+A\sin\delta=6$$

$$\Rightarrow -4+A\sin\delta=6 \Rightarrow$$

$$\begin{matrix} A\cos\delta=2 \\ A\sin\delta=10 \end{matrix} \Rightarrow \begin{matrix} A^2\cos^2\delta=4 \\ A^2\sin^2\delta=100 \end{matrix}$$

$$A^2=104$$

$$A=\sqrt{104}$$

$$\tan\delta=5 \rightarrow \delta=\tan^{-1}(5)=1.373$$

$$\rightarrow y(t)=\sqrt{104}\,e^{-2t}\cos(\sqrt{3}t-1.373)$$

$\Rightarrow$ first time it crosses the equilibrium pt.

$$\Rightarrow \sqrt{3}t-1.373=\begin{cases}\frac{\pi}{2}\\ -\frac{\pi}{2}\end{cases} \Rightarrow t=\begin{cases}\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}+1.373\right)\\ \frac{1}{\sqrt{3}}\left(1.373-\frac{\pi}{2}\right)\end{cases}$$

$$t=\begin{cases}1.699\text{ sec} & \leftarrow \text{Ans.}\\ -0.1142\end{cases}$$

4. Use Laplace Transform to solve: $y''+3y'+2y=12te^{2t}; y(0)=0, y'(0)=1$ (5 pts)

$$\mathcal{L}(y'')+3\mathcal{L}(y')+2\mathcal{L}(y)=12\mathcal{L}(te^{2t})$$

$$s^2\mathcal{L}(y)-s(0)-1+3[s\mathcal{L}(y)-0]+2\mathcal{L}(y)=12\cdot\frac{1}{(s-2)^2}$$

$$\underbrace{(s^2+3s+2)}_{(s+2)}\mathcal{L}(y)=\frac{12}{(s-2)^2}+1=\frac{12+s^2-4s+4}{(s-2)^2}$$

$$\mathcal{L}(y)=\frac{s^2-4s+16}{(s-2)^2(s+2)(s+1)}=\frac{A}{s-2}+\frac{B}{(s-2)^2}+\frac{C}{s+2}+\frac{D}{s+1}$$

$$B\Big|_{s=2}=\frac{4-8+16}{4\cdot3}=1$$

$$C\Big|_{s=-2}=\frac{4+8+16}{16(-1)}=-\frac{7}{4}$$

$$D\Big|_{s=-1}=\frac{1+4+16}{9(1)}=\frac{7}{3}$$

let $s=0 \Rightarrow \frac{16}{4(2)}=\frac{A}{-2}+\frac{1}{4}-\frac{7}{8}+\frac{7}{3}=2$

$$\Rightarrow \frac{A}{2}=\frac{1}{4}-\frac{7}{8}+\frac{7}{3}-2=\frac{2-7+56-48}{24}=\frac{3}{24}=\frac{1}{8}\Rightarrow A=\frac{1}{4}$$

$$\Rightarrow y(t)=\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)+\mathcal{L}\left(\frac{1}{(s-2)^2}\right)-\frac{7}{4}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)+\frac{7}{3}\mathcal{L}\left(\frac{1}{s+1}\right)$$

$$y(t)=\frac{1}{4}e^{2t}+te^{2t}-\frac{7}{4}e^{-2t}+\frac{7}{3}e^{-t}.$$

5. Using Eigenvalues/Eigenvectors to solve: (20 pts)

5a) $\frac{d\overrightarrow{X}}{dt} = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix} \overrightarrow{X(t)}; \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(-1-i)x - y = 0$

pick $x = -1 \Rightarrow y = 1+i$

$\vec{V} = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$

Eigenvalues $\Rightarrow p(\lambda) = \lambda^2 - 6\lambda + 10 = \lambda^2 - 6\lambda + 9 + 1 = 0$

$(\lambda - 3)^2 = -1 \Rightarrow \lambda = 3 \pm i$.

Eigenvectors: $\Rightarrow \begin{bmatrix} 2-(3+i) & -1 \\ 2 & 4-(3+i) \end{bmatrix} \Rightarrow \begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t}\left[\cos t + i\sin t\right] \begin{bmatrix} -1 \\ 1+i \end{bmatrix} = e^{3t} \begin{bmatrix} -\cos t - i\sin t \\ \cos t + i\sin t + i\cos t - \sin t \end{bmatrix}$.

General Sol^{n}: $\vec{X} = e^{3t}\left\{ c_1 \begin{bmatrix} -\cos t \\ \cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \sin t + \cos t \end{bmatrix} \right\}$

$\vec{X}(0) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{array}{l} -c_1 = 1 \Rightarrow c_1 = -1 \\ c_1 + c_2 = -1 \Rightarrow c_2 = 0 \end{array}$

5b) $\frac{d\overrightarrow{X}}{dt} = \begin{bmatrix} 0 & 5 \\ 2 & 3 \end{bmatrix} \overrightarrow{X(t)}; \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \Rightarrow p(\lambda) = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2) = 0$

Eigenvalues $\Rightarrow \lambda = 5, -2$

$\lambda = 5 \Rightarrow \begin{bmatrix} -5 & 5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow -5x + 5y = 0 \Rightarrow \vec{V}_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2 \Rightarrow \begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow 2x + 5y = 0 \left\{ \begin{array}{l} x = 5 \\ y = -2 \end{array} \right. \Rightarrow \vec{V}_{-2} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

$\vec{X} = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

$\vec{X}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$$\begin{array}{rl} & c_1 + 5c_2 = 3 \\ - & c_1 - 2c_2 = -3 \\ \hline & 7c_2 = 6 \end{array}$$

$c_2 = \frac{6}{7}$

$c_1 = -3 + 2\left(\frac{6}{7}\right) = -3 + \frac{12}{7} = -\frac{9}{7}$

Sol^{n}: $\vec{X}(t) = -\frac{9}{7} e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{6}{7} e^{-2t} \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

5c) $\frac{\overrightarrow{dX}}{dt} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{25}{2} & -4 \end{bmatrix} \overrightarrow{X(t)};$ $\Rightarrow p(\lambda) = \lambda^2 + 3\lambda + \left(-4 + \frac{25}{4}\right)$

$= \lambda^2 + 3\lambda + \frac{9}{4} = 0.$

$= \left(\lambda + \frac{3}{2}\right)^2 = 0 \Rightarrow \lambda = -\frac{3}{2}, -\frac{3}{2}$

$\Rightarrow \begin{bmatrix} 1+\frac{3}{2} & -\frac{1}{2} \\ \frac{25}{2} & -4+\frac{3}{2} \end{bmatrix} := \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{25}{2} & -\frac{5}{2} \end{bmatrix} \Rightarrow \frac{5}{2}x - \frac{1}{2}y = 0 \Rightarrow \begin{matrix} x = 1 \\ y = 5 \end{matrix} \Rightarrow \vec{K} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$

$5x - y = 0$

Defective: $(A - \lambda I)\vec{P} = \vec{K}$. ✓

$\Rightarrow \begin{bmatrix} 5/2 & -1/2 \\ 25/2 & -5/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \frac{5}{2}x - \frac{1}{2}y = 1 \Rightarrow$ let $y = 0 \Rightarrow x = \frac{2}{5} \Rightarrow \vec{P} = \begin{bmatrix} 2/5 \\ 0 \end{bmatrix}.$

Sol: $\vec{X} = e^{-3/2 t}\left[c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2\left(\begin{bmatrix} 2/5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)\right].$

5d) $\frac{\overrightarrow{dX}}{dt} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \overrightarrow{X(t)}$ $\Rightarrow p(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$

$\Rightarrow p(\lambda) = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix}$

$= -\lambda\left[\lambda^2 - 1\right] - (-\lambda - 1) + (1 + \lambda).$

$= -\lambda(\lambda - 1)(\lambda + 1) + 2(\lambda + 1)$

$= (\lambda + 1)\left[-\lambda(\lambda - 1) + 2\right]$

$= (\lambda + 1)\left[-\lambda^2 + \lambda + 2\right]$

$= -(\lambda + 1)(\lambda^2 - \lambda - 2)$

$= -(\lambda + 1)(\lambda - 2)(\lambda + 1)$

$\Rightarrow \lambda = -1, -1, 2.$

$\lambda = -1 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x + y + z = 0$

$x = -y - z$

$\vec{V}_{-1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y - z \\ y \\ z \end{bmatrix}$

$= y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

$\vec{V}_{-1} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 2 \Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\begin{cases} -2x + y + z = 0 \\ x - 2y + z = 0 \\ x + y - 2z = 0 \end{cases}$ $\Rightarrow \begin{cases} -3y + 3z = 0 \\ y = z . \end{cases}$

$x + z - 2z = 0 \Rightarrow x = z .$

$$V_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

sol: $\vec{X}(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$