Worksheet #1 **_Math 285_**

1. Consider the problem of forecasting the future size of the population in some country. We find that among every 1000 people alive today, on the average there will be 25 births and 20 deaths in the next year. We also find that, on the average, 3 people out of every 1000 emigrate every year and that a total of 100,000 people immigrate. If the population is 50 million, what will be the population 10 years from now if present trends continue?
Rate of change of the population would be

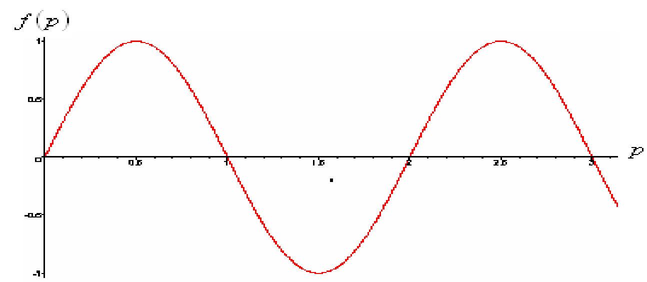
$$\frac{dp}{dt} = \text{birth rate} - \text{death rate} + \text{immigration rate} - \text{emigration rate}.$$

   a) Let p be the present population, then we have the following:
   birth rate = 0.025p; death rate = 0.020p; emigration rate 0.003p and immigration rate = 100,000.
   Set up the DE for the above scenario.
   b) Show that $p(t) = 50{,}000{,}000\left(e^{0.002t} - 1\right) + Ae^{0.002t}$ is a solution for part (a).
   c) Explain what A signifies.
   d) If the current population is 50 million, what will the population be in 20 years?
   e) How does your answer in (d) change if the current population is 100 million? Is the population in 20 years twice as large? Explain.

2. For each of the following equations, find the value(s) of a for which the function $y(x) = e^{ax}$ is a solution.
   a) $y'' = 2y' + 7$ b) $y'' - 4y' + 4y = 0$
   c) $y'' = y$ d) $6y''' - 7y'' - 3y' = 0$

3. Let $p' = t - 2p$.
   a) Describe the behavior of solutions as t gets large.
   b) Show by substitution that $p(t) = At + B$ is a solution for appropriate values of A and B (What are they?)

4. Suppose $\frac{dp}{dt} = f(p)$, where the corresponding phase diagram is shown below. Which of the following statements concerning $p(t)$ are true and which are false? Justif



   a) $p(t) = 1$ for all t.
   b) $p(t)$ is a periodic function.
   c) $p(2) = 2$ then $p(t) = 2$ for all t.
   d) 1 is a stable equilibrium, and 2 is an unstable equilibriu
   e) if $p(0)$ is sufficiently large, then $\lim_{t\to\infty} p(t) = \infty$

5. ***<u>Mixing Problems</u>***
   a) Suppose alcohol is introduced into a 2-liter beaker, which initially contains pure water, at the rate of 0.1 L/min. The well-stirred mixture is removed at the same rate.
      i) How long does it take for the concentration of alcohol to reach 50%? 75%? 87.5 %?
      ii) Suppose the current concentration of water in the beaker is c and we ask how long it takes before the concentration is cut in half. Is this time interval the same, regardless of c?
   b) Suppose a brine containing 2 kg of salt per liter runs into a tank initially filled with 500 L of water containing 50 kg of salt. The brine enters the tank at a rate of 5 L/min. The mixture, kept uniform by stirring, is flowing out at the rate of 5 L/min.
      i) Find the concentration, in kilograms per liter, of salt in the tank after 10 min.
      ii) After 10 min a leak develops in the tank and an additional liter per minute of mixture flows out of the tank. What will be the concentration, in kilograms per liter, of salt in the tank after 20 min?
   c) Blood carries a drug into an organ at a rate of 3 cm^3/sec and leaves at the same rate. The organ has a liquid volume of 125 cm^3. If the concentration of the drug in the blood entering the organ is 0.2 g/cm^3, what is the concentration of the drug in the organ at time t if there was no trace of the drug initially? When will the concentration of the drug in the organ reach 0.1 g/cm^3?

6. ***Newton's Law of Cooling***:

a) A cold beer initially at 35ºF warms up to 40ºF in 3 min while sitting in a room of temperature 70ºF. How warm will the beer be if left out for 20 min?

b) A red wine is brought up from the wine cellar, which is a cool 10ºC, and left to breathe in a room of temperature 23ºC. When will the temperature of the wine reach 18ºC if it takes 10 min for the wine to reach 15ºC?

c) i) Suppose the temperature of the body of a homicide victim is 23ºC when it is discovered in a room whose ambient temperature is 20ºC. If nominal body temperature is 37ºC and the body cools by 2ºC one hour after death, how long ago did the murder occur?

ii) In a celebrated criminal case, former football start O.J. Simpson was accused of murdering his ex-wife Nicole Brown Simpson and her friend Ronald Goldman between 9:45 and 10:30 one evening in June 1994. One of the critical points in the case was to establish the time of the murders more precisely. Newton's law of cooling was not used for this purpose, and the remainder of this exercise explores one reason why.

iii) The rate at which a body cools depends in part on its size and how it is clothed. In any particular case, there is some uncertainty about how much the body might cool in an hour. Suppose that the conditions were as described in (i) but that the bodies cooled by 3ºC one hour. Repeat your calculation of the time of death.

iv) Nominal body temperature varies among individuals. Assume the same set of circumstances as in (i), but suppose that the body temperature of the victims was 36ºC at the time of the murders. How long ago did the murders occurs?

v) On the basis of your experience in this question, how good a forensic tool might Newton's law of cooling be? Consider the range of estimates that you obtain for the time of death as you formulate your answer.

Series Circuits:

a) A 30 – volt electromotive foce is applied to an LR series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms. Find the current i(t) if i(0)=0. Determine the current as t approaches infinity.

b) A 100 – volt electromotive force is applied to an RC series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4}farad. Find the charge q(t) on the capacitor if q(0)=0. Find the current i(t).

c) A 200 – volt electromotive force is applied to an RC series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-4}farad. Find the charge q(t) on the capacitor if $i(0) = 0.4\ amp$. Determine the charge and the current at $t = 0.005 sec$. Determine the charge as $t \to \infty$

d) An electromotive force $E(t) = \begin{cases} 120 & 0 \le t \le 20 \\ 0 & t > 20 \end{cases}$ is applied to an LR series circuit in which the inductance is 20 henries and the resistance is 2 ohms. Find the current $i(t)\ if\ \ i(0) = 0$.

7. Solve the following:

a) $x\dfrac{dy}{dx} + 2y = 5x^3$

b) $(x^2+1)\dfrac{dy}{dx} + xy = x$

c) $x\dfrac{dy}{dx} + 3y + 2x^2 = x^3 + 4x$

d) $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

e) $(x^2 + y^2)dx + 2xydy = 0$

f) $(3x^2 - y^2)dx + (xy - x^3y^{-1})dy = 0$

g) $\dfrac{dy}{dx} = \dfrac{y^2 + x\sqrt{x^2+y^2}}{xy}$

h) $\dfrac{dy}{dx} = \dfrac{y(\ln y - \ln x + 1)}{x}$

i) $(2xy + 3)dx + (x^2 - 1)dy = 0$

j) $(e^x \sin y - 3x^2)dx + (e^x \cos y + y^{-2/3}/3)dy = 0$

k) $\dfrac{dy}{dx} = \dfrac{x+2y-1}{2x-y+3}$

l) $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

8. Solve the following IVP

a) $\frac{dy}{dx} = 2\sqrt{y+1}\cos x;\ y(\pi) = 0$

b) $\frac{dy}{dx} = 8x^3 e^{-2y};\ y(1) = 0$

c) $\frac{dy}{dx} = 2x\cos^2 y;\ y(0) = \pi/4$

d) $\sqrt{y}\,dx + (1+x)dy = 0;\ y(0) = 1$

e) $\frac{dy}{dx} + 4y - e^{-x} = 0;\ y(0) = \frac{4}{3}$

f) $\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x;\ y(1) = 1$

g) $(e^x y + 1)dx + (e^x - 1)dy = 0;\ y(1) = 1$

h) $(y^2 \sin x)dx + (1/x - y/x)dy = 0;\ y(\pi) = 1$

9. ***Constant Multiples of solutions***.

a) Show that $y = e^{-x}$ is a solution of the linear equation $\frac{dy}{dx} + y = 0$ and $y = x^{-1}$ is a solution to the nonlinear equation $\frac{dy}{dx} + y^2 = 0$

b) Show that for any constant C, Ce^{-x} is a solution of equation $\frac{dy}{dx} + y = 0$, while Cx^{-1} is a solution of equation $\frac{dy}{dx} + y^2 = 0$ only when $C = 0$ *or* 1

c) Show that for any linear equation of the form $\frac{dy}{dx} + P(x)y = 0$ if $\hat{y}(x)$ is a solution, then for any constant C, the function $C\hat{y}(x)$ is also a solution.

10. Determine whether the Existence and Uniqueness theorem implies that the following IVP has a unique solution.

a) $\frac{dy}{dx} = x^3 - y^3;\ y(0) = 6$

b) $\frac{dy}{dx} - xy = \sin^2 x;\ y(\pi) = 5$

c) $\frac{dy}{dx} + \cos y = \sin x;\ y(\pi) = 0$

d) $y\frac{dy}{dx} - 4x = 0;\ y(0) = 0$

11. ***Uniqueness Questions***:
When we use method of separation of variable to solve a DE, what if the DE does not satisfy the Existence and Uniqueness Theorem, then the solution from separation of variables may not give us all the solutions of that DE. To see this, consider the equation $dy/dx = y^{1/3}$

a) Use the method of separation of variables to show that $y = \left(\frac{2x}{3} + C\right)^{3/2}$ is a solution.

b) Show that the initial value problem $dy/dx = y^{1/3}$ with $y(0) = 0$ is satisfied for C = 0 by $y = (2x/3)^{3/2}$

c) Now show that the constant function $y \equiv 0$ also satisfies the IVP given in part (b). Hence this initial value problem does not have a unique solution.

d) Finally, show that the condition of the Existence and Uniqueness Theorem are not satisfied.

12. ***Division by Zero***.
In developing our method of separation of variables for $dy/dx = f(x)g(y)$, we tacitly assumed, when we divided by $g(y)$ to obtained $dy/g(y) = f(x)dx$, that $g(y) \neq 0$. This assumption may cause us to lose solutions. Let's examine the following:

a) For the equation $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$, use separation of variables to derive the solution

$y = -1 + (x^2/6 - x + C)^3$

b) Show that $y \equiv -1$ satisfies the original equation $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$

c) Show that there is no choice of the constant C that will make the solution in part (a) yield the solution $y \equiv -1$. Thus we lost the solution $y \equiv -1$ when we divided by $(y+1)^{2/3}$

13. Find vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $\vec{b}$ satisfies the equation $A\vec{v} = \vec{b}$ where $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

a) $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 9 & 1 & 1 \\ -1 & 3 & 4 \\ -2 & -1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \\ -9 & 3 \end{bmatrix}$

14. Let $A = \begin{bmatrix} a & x \\ x & 3x \end{bmatrix}$

a) For which number x will A be singular?

b) For all number x not on your list in part (a), we can solve $A\vec{v} = \vec{b}$ for every vector $\vec{b} \in R^3$. For each of the numbers x on your list, give the vector $\vec{b}$ for which we can solve $A\vec{v} = \vec{b}$